Tenney-Euclidean Formulas

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January 28, 2011

Simple Formulas

Error

The Tenney-Euclidean (TE) or Tenney-weighted, optimal, prime-based, root mean squared (TOP-RMS) error is a way of measuring the overall error of an optimal tuning of a temperament simply by looking at the errors of its approximations to the prime numbers. It was inspired by Paul Erlich's TOP, which does the same thing. But TOP makes the largest weighted error over all intervals as small as possible (minimax; L1 norm) whereas TE uses the RMS error over the prime intervals (least squares; L2 norm). It's naturally dimensionless but you can multiply by 1200 to get units of cents/octave.

You can compute the TE scale stretch of an equal temperament given a mapping from primes to generators (val) with this formula:

$$\sum_i v_i / \sum_i v_i^2, \qquad v_i = \frac{m_i}{\log_2(p_i)} \tag{1}$$

Here, p_i is the ith prime number and m_i is the number of scale steps you approximate it with in the equal temperament. To get a step size in cents, multiply this value by 1200 and divide it by the number of steps to the octave.

The dimensionless TE error of the same equal temperament is

$$\sqrt{\frac{\sum_i v_i^2 - \left(\sum_i v_i\right)^2/n}{\sum_i v_i^2}} \tag{2}$$

where p_i is the ith prime number, m_i is the number of scale steps you approximate it with in the equal temperament, and n is the number of primes under consideration.

A TE generator of a rank 2 temperament given two defining vals is

$$g_1 = 1200\frac{\sum_i u_i \sum_i v_i^2 - \sum_i v_i \sum_i u_i v_i}{\sum_i u_i^2 \sum_i v_i^2 - \left(\sum_i u_i v_i\right)^2} \tag{3}$$

$$u_i = \frac{m_{1i}}{\log_2(p_i)}, \qquad v_i = \frac{m_{2i}}{\log_2(p_i)} \tag{4}$$

where p_i is the ith prime, m_{1i} is the number of steps of one generator corresponding to p_i, and m_{2i} is the number of steps of the other generator corresponding to p_i. The formula gives the size of the first generator (m_1 or u) in cents. To get the other generator, swap u and v.

The dimensionless TE error of the same temperament is

$$n\sqrt{\frac{\operatorname{cov}(u,u)\operatorname{cov}(v,v) - [\operatorname{cov}(u,v)]^2}{\sum_i u_i^2 \sum_i v_i^2 - \left(\sum_i u_i v_i\right)^2}} \tag{5}$$

where u, v, and n are as above and the covariance is defined as

$$\operatorname{cov}(x,y) = \sum_{i=1}^{n} \frac{x_i y_i}{n} - \sum_{i=1}^{n} \frac{x_i}{n} \sum_{i=1}^{n} \frac{y_i}{n}. \tag{6}$$

Complexity

TE complexity is a way of measuring the number of notes you need in a scale in order to map intervals from just intonation. It uses the same weighting as the TE error so that intervals are considered according to the complexity of their ratios. The units are steps$^{\mathrm{r}}$/octaves^r for a rank r temperament.

You can compute the TE complexity of an equal temperament given a val as

$$\sqrt{\sum_i \frac{m_i^2}{n[\log_2(p_i)]^2}} \tag{7}$$

where p_i is the ith prime number, m_i is the number of scale steps you approximate it with in the equal temperament, and n is the number of primes under consideration. For sensible equal temperaments, TE complexity is almost exactly the number of steps to an octave.

You can compute the TE complexity of a rank 2 temperament given a pair of vals as

$$\sqrt{\sum_i \frac{u_i^2}{n} \sum_i \frac{v_i^2}{n} - \left(\sum_i \frac{u_i v_i}{n}\right)^2} \tag{8}$$

where u and v are weighted vals as defined in equation 4 and n is the number of primes being used in the ratios you're interested in.

Cangwu Badness

Badness is a quantity that gives you an idea of how efficient a temperament is – how good the harmony is for a given number of notes. Cangwu badness is so named because I thought of it in Cangwu Park in Lianyungang. Most badness measures are a function of error and complexity that increases as either error or complexity increase. For an equal temperament, Cangwu badness works like this. You can compute it given TE complexity k and error E as

$$k\sqrt{E^2 + E_k^2} \tag{9}$$

where E_k is a free parameter that roughly corresponds to the smallest error you care about. A larger E_k makes temperaments with higher error come out better.

The Cangwu bandess of a rank 2 temperament is

$$\sqrt{\mathrm{cov}(E_k, u, u)\mathrm{cov}(E_k, v, v) - [\mathrm{cov}(E_k, u, v)]^2} \tag{10}$$

where u and v are weighted vals as in equation 4 and a modified covariance is defined as

$$\mathrm{cov}(E_k, x, y) = (1 + E_k^2)\sum_{i=1}^{n} \frac{x_i y_i}{n} - \sum_{i=1}^{n} \frac{x_i}{n} \sum_{i=1}^{n} \frac{y_i}{n}. \tag{11}$$

Advanced Formulas

Symbols

I'm supposed to be avoiding spaghetti definitions, but I'll define the three matrices these formulas use anyway.

M is the mapping of primes to generators, with vals as rows.

H is a row vector of JI prime intervals in octaves. For consecutive prime limits,

$$H_i = \log_2(p_i) \tag{12}$$

where p_i is the ith prime number.

W is a matrix that determines the weighting. For Tenney weighting, diagonal entries are reciprocals of prime intervals:

$$W_{ii} = \frac{1}{H_i} \tag{13}$$

and off-diagonal entries are zero:

$$W_{ij} = 0, \quad i \neq j. \tag{14}$$

Note that every entry of HW is equal to one. Other weightings are possible. Here, I'll assume W is symmetric, so that $W^T W$ can be written as W^2.

Complexity

$$\sqrt{\det\left[\frac{MW^2M^T}{HW^2H^T}\right]} \tag{15}$$

Error

This is only TE for Tenney weighting, but gives the optimal RMS error for any weighting. The general problem is a weighted least squares optimization where the error is defined as

$$\sqrt{(XMW - HW)(XMW - HW)^T} \tag{16}$$

with X as a column matrix giving the generator sizes in octaves. For the optimal generators, use your favorite least squares algorithm, and note that

$$X_{\text{opt}} = HW^2M^T(MW^2M)^{-1}. \tag{17}$$

You can also compute the optimal error without finding the optimal tuning:

$$\sqrt{\frac{\det\left[\frac{MW^2M^T}{HW^2H^T}\right]}{\det\left[\frac{MW^2M^T}{HW^2H^T} - \frac{MW^2H^THW^2M^T}{(HW^2H^T)^2}\right]}}. \tag{18}$$

Cangwu Badness

$$\sqrt{\det\left[\frac{MW^2M^T}{HW^2H^T}(1+E_k^2) - \frac{MW^2H^THW^2M^T}{(HW^2H^T)^2}\right]} \tag{19}$$

E_k is a free parameter as above.

Multilinear Algebra

You may choose to define a temperament using a wedgie. If you don't know what a weighted wedgie is, best to skip this section. Given that weighted wedgie T, TE complexity is

$$||T|| \tag{20}$$

where the norm $||\ldots||$ can be defined the square root of either as the absolute value of the inner product of the weighted wedgie with itself, or sum of the squares of the entries of the linearized weighted wedgie. To agree with the formula given above, for Tenney weighting, you need to divide the weighted wedgie by the number of primes. In general, this gives

$$||(T/||J||)|| \tag{21}$$

where $J = HW$ is the weighted JI point.

The TE error is

$$\frac{||T \wedge J||}{||T||\,||J||} \tag{22}$$

Alternative Weightings

If you want to use an equally weighted RMS over a set of intervals, write them as a matrix C where each row is an interval in vector form. Then, substitute

$$W^2 = C^TC \tag{23}$$

in the formulas above.

Geometry

TE complexity corresponds to a Euclidean distance in a vector space with a metric determined by the weighting matrix. The TE error is the sine of the angle in this space between the line joining the origin to the JI point and the flat (line, plane, etc.) including the origin and all vals belonging to the temperament.

With a reasonable weighting, including Tenney weighting, Cangwu badness is the equivalent of TE complexity in an *inner product space*. It is a positive definite quadratic form.

Background Reading

For background, see http://x31eq.com/primerr.pdf