Prime Based Error and Complexity Measures

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When comparing different temperaments or temperament families, it's useful to know how closely they approximate just intonation. There are different ways of doing this. In the absence of sound, empirical data to tell us what measures work best, the safest thing is to choose the simplest one that works reasonably well. Finding it is surprisingly difficult.

I assume you have some background in both mathematics and tuning theory. The mathematics isn't very advanced but there are a lot of equations. Partly this is to make it easier to understand by not skipping intermediate steps. But you should still note that there's far more detail here than most people are ever going to need. So it's for people who aren't afraid of mathematics, but aren't necessarily experts in it either.

The original ideas here arose from discussions on the tuning-math mailing list at *Yahoo! Groups*, `http://groups.yahoo.com/group/tuning-math` and related lists. I also had a private e-mail discussion with William Sethares and some real world conversations with members of the mathematics department of Huaihai Institute of Technology, Lianyungang.

Contents

1 Basic Ideas

1.1 Qualitative Properties of Tuning Error

Before leaping into a quantitative discussion of the errors of different temperaments, it may be worth thinking about what it means for a tuning to have an error. Error relative to what, and why should we care? Without surveying the relevant literature (which is far from conclusive anyway) I hope we can agree on the following qualitative properties.

1. Simple ratios have a desirable affect.
2. The closer to a ratio the stronger the affect.
3. Moderate errors stand out more than small ones.
4. Large errors are irrelevant.

The first point may appear obvious – of course we prefer simple ratios! Then again, you may instantly object to it. If so, I don't see any point in arguing with you. If you're undecided, listening to examples (ideally of your own making) will be more persuasive than any arguments I could make, or any authorities I could quote. If your conclusion is that one interval's as good as another you can save yourself some trouble by stopping reading here.

I'll leave aside for now the problems of inharmonic timbres, or preference for stretched or compressed harmonic series. To an extent the harmonic series is represented by free parameters in these models, and you can tweak the parameters if the harmonic series isn't what you wanted. I'll deal with this when the time is right. For now, let's assume that the true ratios really are the ones we want because it makes the descriptions easier.

I hope the second point is obvious. Note that it doesn't state that the best tuning is always the exact one. Only that the affect of true ratio-ness will be stronger the closer you get to a true ratio. That doesn't mean that the desirable affect becomes more desirable; you can have too much of a good thing.

The third point says that we'll tolerate small mistunings a lot more than moderate ones. Where you draw the line between "small" and "moderate" is up to you. But if one note in a chord or melody is much more out of tune than the others I hope you'll agree that it's the one you'll notice the most.

Finally, I'm simply observing that if a mistuning gets stupidly large, an interval won't be heard as an approximation of the ratio you're measuring it relative to. As an extreme example, calling a semitone an approximation to 3:2 is meaningless. Saying that a whole tone has a smaller error as a 3:2 is also meaningless. It's better to say that neither of them will sound at all like a 3:2.

1.2 Tuning of Prime Limits

If a tuning system approximates a prime limit, the tuning of the approximation to any interval within the prime limit can be determined if you know the tuning of the intervals that approximate the prime numbers. By "prime intervals" I usually mean prime number ratios or approximations thereof. For conventional just intonation, the prime intervals are 2:1, 3:1, and 5:1, and so on. I use the coefficients of the prime factorization, x_i, of the frequency ratio n:d such that

$$\frac{n}{d} = \prod_i p_i^{x_i} \tag{1}$$

where p_i is the ith prime number.

The size of an interval measured in log-frequency units is

$$s(x) = \sum_i x_i h_i \tag{2}$$

where h_i are the prime intervals in whatever log-frequency units you choose to use. For 5-limit just intonation,

$$\begin{aligned} h_0 &= \log_2(2)\text{octaves} \\ h_1 &= \log_2(3)\text{octaves} \\ h_2 &= \log_2(5)\text{octaves} \end{aligned}$$

(I count my indexes from zero so that h_0 is the octave. Sometimes you can work in octave-equivalent coordinates, and then you can ignore octaves and start the indexes from one instead. If you don't like this convention, don't worry too much.) If you prefer cents,

$$\begin{aligned} h_0 &= 1200\log_2(2)\text{cents} \\ h_1 &= 1200\log_2(3)\text{cents} \\ h_2 &= 1200\log_2(5)\text{cents} \end{aligned}$$

1.3 Regular Temperament Errors

In a regular temperament, each prime interval h_i takes on a tempered value t_i and the size of the tempered interval is

$$s(x) = \sum_i x_i t_i \tag{3}$$

The deviation of an interval relative to just intonation is the difference between the sizes in equations 2 and 3 on the preceding page :

$$\begin{aligned} d(x) &= \sum_i (x_i t_i - x_i h_i) \\ &= \sum_i x_i (t_i - h_i) \\ &= \sum_i x_i d_i \\ d_i &= t_i - h_i \end{aligned} \tag{4}$$

with d_i as the deviation of the ith prime interval. This is defined such that the deviation will be positive for an interval that's sharp of just intonation. For example, the prime deviations of 12 note equal temperament of 5-limit just intonation are

$$\begin{aligned} d_0 &= 1200 - 1200\log_2(2)\text{cents} \\ d_1 &= 1900 - 1200\log_2(3)\text{cents} \\ d_2 &= 2800 - 1200\log_2(5)\text{cents} \end{aligned}$$

The error in an interval is the absolute value of its deviation from just intonation.

1.4 Prime Weighting

Each interval in a regular temperament has a unique error associated with it. All intervals can be derived from the prime intervals. So, a simple way of assessing the error of a temperament is to do some kind of average of the errors of the prime intervals.

The simplest approach is to treat all prime errors equally. This gives unreasonable results in practice. For example, the intervals 2:1 and 7:1 are counted on an equal footing, so an error of 1 cent in 2:1 is treated as badly as an error of 1 cent in 7:1. But the error in 8:1 has to be three times as big as the error in 2:1 for any regular temperament. So 8:1 is allowed to be three times as out of tune as 7:1 when it's only a little bit bigger! To give more flexibility, each prime error is given a different weighting when calculating the overall error for a temperament.

Instead of weighting the primes, you could take all the intervals you want to use harmonically and calculate the unweighted error for each. This is perfectly valid, and it's what most theorists have done in the past, and will probably continue to do. One advantage of weighted prime errors is that they can be evaluated with less calculations. They also allow you to be a little bit vague about which intervals you're interested in. So today we'll be talking about weighted prime errors.

The general formula for a weighted prime error is

$$e_i = \left| \frac{t_i}{b_i} - \frac{h_i}{b_i} \right| \tag{5}$$

where e_i is the weighted error of the ith prime interval and b_i is positive real number determining the weight given to that interval. The larger b_i is, the less weight the corresponding interval has. That means it's more of a buoyancy factor than a weighting factor.

Equation 5 can be re-written

$$\begin{aligned} e_i &= |w_i - v_i| \\ v_i &= \frac{h_i}{b_i} \\ w_i &= \frac{t_i}{b_i} \end{aligned} \tag{6}$$

with v_i as the weighted size of the ith prime interval in just intonation, and w_i is the weighted size of the ith prime tempered interval, or the ith weighted prime for short. This is the key equation that all subsequent error measures will use.

There are some properties that all prime-weighted error measures of the kind I'll be looking at share, like them or not.

1. More complex intervals are given a lower weight.
2. You can't choose which composites to consider, and the weighting of composites is implied by the weighting of primes.
3. The error is symmetrical as to sharp and flat deviations.

In many cases it's certainly *not* appropriate to give complex intervals a higher weight. You may decide that more complex intervals are harder to hear, and so must be tuned more precisely to help the ear. But in that case you *must* consider a finite set of intervals. Unbounded prime-limit errors will end up being determined by the infinite number of infinitely complex – and therefore completely meaningless – ratios.

The lack of freedom in choosing the weights might be a problem if you have a precise idea what you want the weights to look like. I don't. It's a difficult question to answer in the general case so simplicity wins.

The ear doesn't hear sharp and flat mistunings as equally bad. But I'm assuming there's no way to know if a given prime is more likely to be in the numerator or denominator of ratios. Perhaps you could assume that higher primes are more likely to be in the numerator, and construct an asymmetric prime-based measure accordingly. That might be valid but I haven't considered

it. If you prefer sharp mistunings you can also optimize for symmetry and then stretch the scale a bit.

The buoyancy of a composite interval (being an interval composed of prime intervals) is taken to be the sum of the buoyancies of all the prime intervals that make it up. That's something you have to consider when choosing the prime intervals. A prime interval should be considered more consonant than any of the composite intervals that require it. Usually they will be the strongest independent partials, relative to the fundamental, of the timbre you want to base your harmony on.

It's assumed that composite partials will tend to be weaker than prime intervals above the fundamental. So, 6:1 must be weaker than either 3:1 or 2:1, 10:1 must be weaker than either 5:1 or 2:1, and so on. For harmonic timbres in general, this is reasonable because high harmonics tend to be weaker than low ones, and the weighting will usually be chosen accordingly. For any specific timbre, the weights will almost certainly add up wrongly, so prime weighting can only go so far in supporting specific timbres.

Inharmonic timbres may not have composite partials at all. In this sense, prime based errors are biased towards harmonic timbres. Ideally, you'd only consider intervals between prime intervals for timbres where all the partials are prime intervals. In practice, it doesn't matter much, because none of this is based on accurate psychoacoustics anyway. Feeding in the strongest partials of an inharmonic timbre should give you some idea of which temperaments will work well.

Following Barlow's "harmonicity"[1], and no doubt other precedents[2], we can measure the weight of a composite interval by calculating the unweighted error of the interval and weighting it using the total buoyancy of the prime factors.

$$e(x) = \frac{\left|\sum_i x_i d_i\right|}{\sum_i |x_i| b_i} \tag{8}$$

[1] Harmonicity follows from an "indigestibility" of primes of

$$b_i = 2\frac{(p_i - 1)^2}{p_i} \tag{7}$$

with p_i as the ith prime. It can be considered as an example of a buoyancy factor (Barlow 1987).

[2] There are a lot of papers showing equal temperament errors out there so I don't plan to survey them all. Darreg & McLaren 1991 use weighting for this purpose. The buoyancy of an interval n:d is $n + d$. This doesn't add up the same way as my prime-based buoyancy. They average errors over a selection of composite ratios.

1.5 TOP Errors

Today, TOP is an acronym for "Tenney Optimal Prime"[3] It corresponds to the special case where the buoyancy of each prime interval is equal to the size of that interval in just intonation. That is, $b_i = h_i$. This is called Tenney weighting. It means equation 6 on the preceding page becomes

$$e_i = |w_i - 1| \tag{9}$$

Hence the error is simply a function of the weighted primes.

This treats all numbers on an equal footing, in so far as an error measure that only considers primes can. If you use inharmonic timbres, so that prime intervals do not correspond to prime numbers, this simplification is less justified. Because harmonic timbres are an extremely important special case I'll concentrate on TOP anyway.

You can see the effect of Tenney weighting by comparing Figures 1 and 2 on the next page. Some complex intervals like 9:8 and 8:7 have a large deviation, but the weighting makes this less important. However, the high error in 7:1 dominates the weighted errors when octaves are kept pure.

To get a true TOP error, you have to do some kind of optimization. That should at least involve optimal tempering of the octaves. The simplest prime-based errors don't work properly unless you temper the octaves because the smallest intervals with any given weight are likely to be the most musically significant.

For inharmonic timbres, you may sometimes get a low partial that's so weak that it isn't going to be as important as some larger intervals in the harmony. So, you have three choices. You can ignore the problem and note that some low scored temperaments that approximate this interval might really be better than they look. You can remove this prime interval from consideration altogether. Or, you can replace it with a larger interval formed by adding it to the strongest prime interval. Depending on the context, that may be more trouble than setting a more reasonable weighting in the first place.

The error in Equation 9 is a dimensionless quantity. Some people think about Tenney weighting as always using base two logarithms, even when intervals are measured in cents. That means the error comes out in units of cents per octave. Because the dimensionless error is quite a small number, using cents per octave

[3] The term "TOP" was introduced by Erlich 2006 to mean either "Tenney OPtimal" or "Tempered Octaves Please!" My usage is consistent with both these expansions, however Paul specifically uses it to mean what I call the TOP-max error.

Figure 1: Unweighted Deviations for 19 Tone Equal Temperament

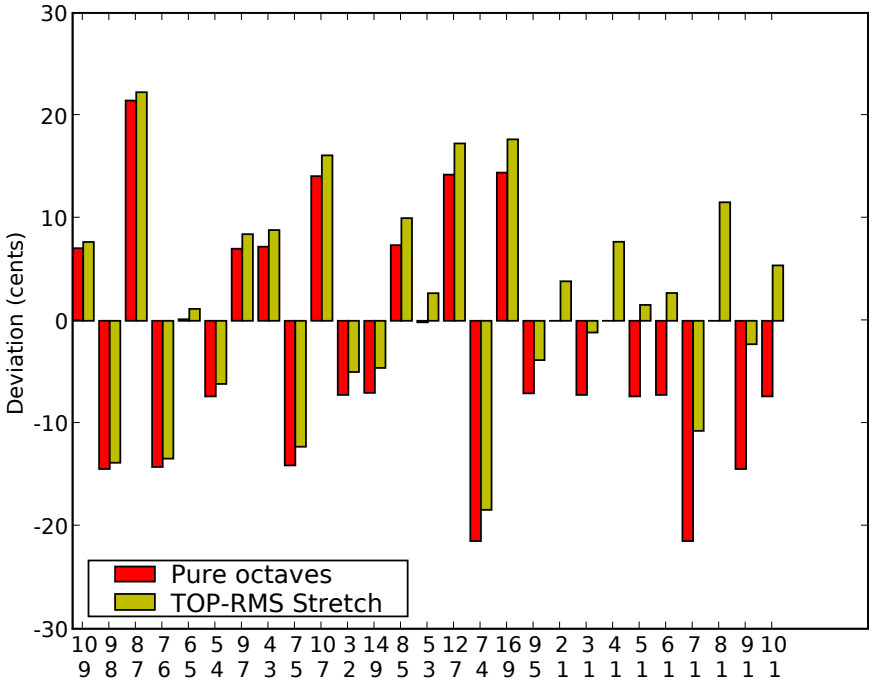
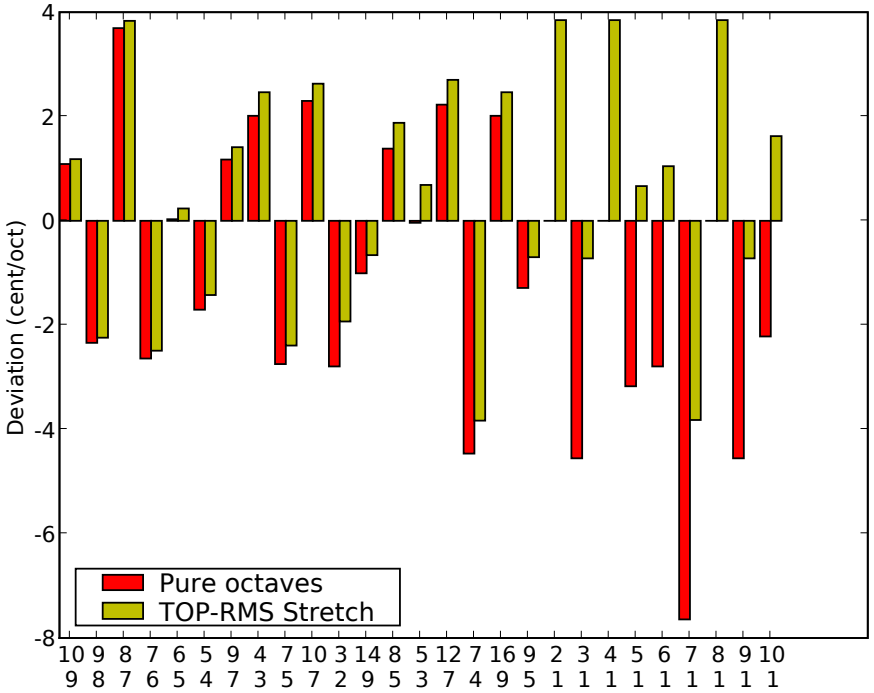


Figure 2: Tenney Weighted Deviations for 19 Tone Equal Temperament



instead is a good practical idea. All you have to do is multiply the dimensionless error by 1200. To make the equations easier, I'll stick with dimensionless error today, but multiply it by 1200 to get cents per octave for the tables.

1.6 Weighted Mappings

I explain mappings for regular temperament classes in Breed 2006. Here, the mapping is a matrix where each row corresponds to the mapping for a prime interval.

The mapping can be weighted the same way that prime intervals can. The formula is

$$m_{ij} = \frac{n_{ij}}{b_i} \tag{10}$$

where n_{ij} is the mapping for the ith prime and jth generator, m_{ji} is the corresponding element of the weighted mapping, and b_i is the buoyancy of the ith prime interval.

The mapping carries all the information about a regular temperament class. Similarly, the errors and complexities I'll be explaining are functions of the weighted mapping.

2 Weighted RMS Errors

2.1 RMS Error of a Temperament

The simplest way of calculating the overall error for a temperament is to take some kind of weighted average of the prime errors. Because the error is the absolute value of the deviation, a simple average is the root mean squared (RMS).

Although the RMS error is the easiest to calculate, that isn't of any value if it doesn't have the properties we're looking for. So let's check that it agrees with what I asked for above.

1. Simple ratios are composed of a small number of prime factors, so an RMS error of the primes will also reflect the error of simple intervals.

2. The RMS error of a set of intervals gets smaller as any interval gets closer to just. An RMS of primes only approximates this property.

3. The RMS error is larger the higher the mistuning, so the more out of tune an interval the more it dominates the average.

4. The RMS will still pay attention to stupidly large mistunings. Because of this you have to choose a sensible mapping before calculating the errors.

So the RMS has the properties we want provided it doesn't get too large. It may not be a true reflection of perceived error but we may as well stick with it until we know what is. The main property of a squared error is that it's symmetrical. As prime-weighted errors are always symmetrical in this way the RMS can't be that far from optimal.

It's easier to show the mean squared error in these equations. All you have to do is take the square root to get the RMS. There's no need to keep showing you all those square root signs. So, this is the general formula for the weighted, mean squared error.

$$\left\langle e^2 \right\rangle = \left\langle (w - v)^2 \right\rangle \tag{11}$$

Where $\langle x \rangle$ is the mean value of x_i over all i.

An alternative is to show the weighted, sum-squared error using matrices.

$$E^T E = (W - V)^T (W - V) \tag{12}$$

Here E, V, and W are column matrices corresponding to the things notated with small letters and subscripts before. That gives the weighted error for a given temperament.

A unification of these two views is to take a weighted sum-squared error normalized by the sizes of the targets.

$$\frac{E^T E}{V^T V} = \frac{(W - V)^T (W - V)}{V^T V} \tag{13}$$

That can be made to look more like Equation 11 by defining a normalized mean squared function

$$\left\langle A^2 \right\rangle_V = \frac{A^T A}{V^T V} \tag{14}$$

For Tenney weighting, this is identical to the mean squared function because all elements of V are 1. The normalized mean square error is then

$$\left\langle E^2 \right\rangle_V = \left\langle (W - V)^2 \right\rangle_V \tag{15}$$

2.2 Optimal Scale Stretching

We can make the calculation more general by allowing for a uniform scale stretch α.

$$\left\langle E^2 \right\rangle_V = \left\langle (\alpha W - V)^2 \right\rangle_V \tag{16}$$

Then optimize α to give the smallest square error.

$$\begin{aligned} \frac{\mathrm{d}E^2}{\mathrm{d}\alpha} &= \frac{\mathrm{d}(\alpha W - V)^T (\alpha W - V)}{\mathrm{d}\alpha} \\ &= 2W^T(\alpha W - V) = 0 \\ \alpha W^T W &= W^T V \\ \alpha &= \frac{W^T V}{W^T W} \end{aligned} \tag{17}$$

Figure 3: Tenney Weighted Prime Deviations for 19 Tone Equal Temperament

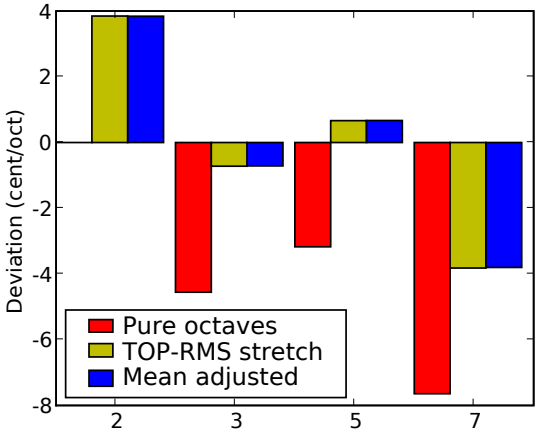
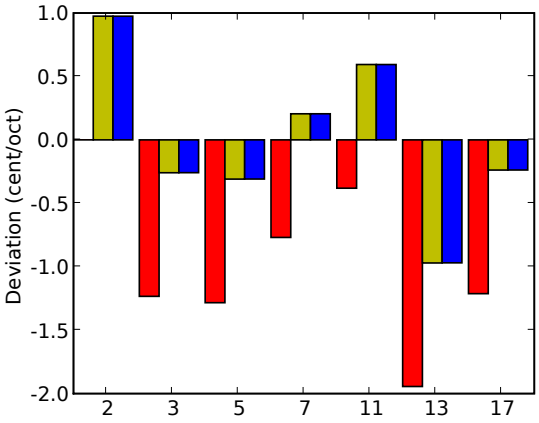


Figure 4: Tenney Weighted Prime Deviations for 72 Tone Equal Temperament



Alternatively,

$$\alpha = \frac{\sum_i w_i v_i}{\sum_i w_i^2} \tag{18}$$

Or, for a TOP error ($v_i = 1$ for all i)

$$\alpha = \frac{\sum_i w_i}{\sum_i w_i^2} \tag{19}$$

Which tells us that the TOP-RMS scale stretch is the mean weighted error divided by the mean squared weighted error, whether the weighted error is optimized or not. You can also use means instead of sums.

$$\alpha = \frac{\langle w \rangle}{\langle w^2 \rangle} \tag{20}$$

To use this form for other weightings, I'll define a normalized mean

$$\langle A \rangle_V = \frac{V^T A}{V^T V} \tag{21}$$

to give

$$\alpha = \frac{\langle W \rangle_V}{\langle W^2 \rangle_V} \tag{22}$$

The second set of bars in Figures 1 and 2 on page 5 show the errors for 19 note equal temperament after it's been stretched by this amount. You can see that it evens out the weighted errors in different intervals. It also makes the worst intervals less bad, so the weighted error of 7:1 is comparable to other weighted errors after the stretch.

19-equal is the classic example of an equal temperament that requires stretched octaves to look at its best. For exact octaves, the deviations of 3:1, 5:1, and 7:1 are all negative. That means that the errors tend to cancel out for the intervals between primes. So, if you look at the weighted errors for pure octaves, the highest values tend to be for primes or powers of primes. Averaging out these values gives an incorrect guess for the average weighted error of smaller, more complex intervals. With the TOP-RMS stretch, the weighted prime errors are more representative of the errors of intervals in general.

Figure 3 shows the effect of optimal stretching on the prime errors of 19-equal. Before stretching, the weighted error of $7:1$ is much worse than simple intervals as a whole (compare with Figure 2 on page 5. After stretching, the prime errors are more moderate and roughly balance each other. The worst prime error is obviously a lot smaller with stretched octaves.

Figure 4 shows the same thing happening for 72-equal in the 17-limit. The different bars have the same meaning as in Figure 3.

Substituting the result of Equation 22 into Equation 16 on the previous page gives

$$\left\langle E_{\text{opt}}^2 \right\rangle_V = \left\langle \left(\frac{\langle W \rangle_V}{\langle W^2 \rangle_V} W - V \right)^2 \right\rangle_V \tag{23}$$

Expanding it all out gives

$$\begin{aligned} \left\langle E_{\text{opt}}^2 \right\rangle_V &= \frac{\left(\frac{\langle W \rangle_V}{\langle W^2 \rangle_V} W - V \right)^T \left(\frac{\langle W \rangle_V}{\langle W^2 \rangle_V} W - V \right)}{V^T V} \\ &= \frac{\left(\frac{\langle W \rangle_V}{\langle W^2 \rangle_V} W^T - V^T \right) \left(\frac{\langle W \rangle_V}{\langle W^2 \rangle_V} W - V \right)}{V^T V} \end{aligned}$$

$$= \frac{\langle W\rangle_V^2}{\langle W^2\rangle_V^2}\frac{W^TW}{V^TV} - \frac{\langle W\rangle_V}{\langle W^2\rangle_V}\frac{W^TV}{V^TV} - \frac{\langle W\rangle_V}{\langle W^2\rangle_V}\frac{V^TW}{V^TV} + \frac{V^TV}{V^TV}$$

Because V and W are both column vectors, W^TV and V^TW are the same, and

$$\begin{aligned}\left\langle E_{\text{opt}}^2\right\rangle_V &= \frac{\langle W\rangle_V^2}{\langle W^2\rangle_V^2}\langle W^2\rangle_V - 2\frac{\langle W\rangle_V}{\langle W^2\rangle_V}\langle W\rangle_V + 1\\ &= \frac{\langle W\rangle_V^2}{\langle W^2\rangle_V} - 2\frac{\langle W\rangle_V^2}{\langle W^2\rangle_V} + 1\\ &= 1 - \frac{\langle W\rangle_V^2}{\langle W^2\rangle_V}\\ \left\langle E_{\text{opt}}^2\right\rangle_V &= \frac{\langle W^2\rangle_V - \langle W\rangle_V^2}{\langle W^2\rangle_V} \qquad (24)\end{aligned}$$

And, for the TOP-RMS error,

$$\sqrt{\left\langle e_{\text{opt}}^2\right\rangle} = \sqrt{\frac{\langle w^2\rangle - \langle w\rangle^2}{\langle w^2\rangle}} \qquad (25)$$

This is the same as the standard deviation of the weighted primes divided by the RMS of the weighted primes.

$$\sqrt{\left\langle e_{\text{opt}}^2\right\rangle} = \frac{\sigma_w}{\sqrt{\langle w^2\rangle}} \qquad (26)$$

2.3 Equal Temperaments

You can see some equal temperament examples in Tables 1 on the following page and 2 on page 10. They show an example for every number of notes to the octave from 5 to 31. The mapping I chose is the one that gives the best TOP-RMS error. A few octave divisions are missing from Table 1 because their mappings all have even numbers in them, and so are effectively the same as an equal temperament with half the number of notes.

I said before that 19-equal is an example of a temperament that needs an octave stretch to be seen in its best light. From Table 1, you can see that it has a 5-limit TOP-RMS error of 1.9 cents per octave, which is simpler than 22-equal, which comes in at 2.7 cents per octave. However, the Tenney-weighted RMS for 19-equal with pure octaves is 3.2 cents per octave whereas 22-equal's is 2.8 cents per octave. In the same way, 19-equal has a Tenney-weighted RMS error comparable to that of 24-equal (4.7 cents per octave against 4.5 for this particular mapping, remember it's inconsistent in the 7-limit) but only if you neglect to optimize the octaves. 19-equal's TOP-RMS error (2.8) is much smaller than 24-equal's (4.5).

There are still a lot of equal temperaments with a larger TOP-RMS scale stretch than that of 19-equal. Although 19-equal has all its 7-limit deviations in the same direction, they're still relatively small. 6-equal has much larger weighted errors, and the deviations are always negative, so it ends up with a scale shrinkage that's about 5 times the stretch of 19-equal.

For good temperaments, the 0.3% scale stretch of 19-equal is about as large as it gets. That means that the optimal scale stretches for good temperaments aren't as big as the stretched tunings used on pianos, or some psychoacoustically reported stretches[4]. As Figure 1 on page 5 shows, the smaller intervals (which are also likely to be the most harmonically important because they stand out the most) are hardly affected by the optimal stretch. Perhaps in some cases you can hear the differences, but you'll have a hard time finding an instrument that reproduces them correctly. So 19-equal is almost as good as the TOP-RMS value says it should be, even without the stretching. The stretch is as much a theoretical device to get the correct error as an instruction for optimal tuning.

2.4 Higher Rank Temperaments

For the more general case, we need to define a weighted mapping M and generators G such that $W = MG$. Equation 12 on page 6 then becomes

$$E^2 = (MG - V)^T(MG - V) \qquad (27)$$

Optimizing it for a small error gives

$$\begin{aligned}\frac{\mathrm{d}E^2}{\mathrm{d}G} &= 2M^T(MG_{\text{opt}} - V) = 0\\ M^TMG_{\text{opt}} &= M^TV\\ G_{\text{opt}} &= (M^TM)^{-1}M^TV \qquad (28)\end{aligned}$$

where $(M^TM)^{-1}$ is the inverse of the square array produced by a matrix multiplication of M with its transpose.[5]

There's also a simplified form of the sum squared error at the optimal point.

$$E_{\text{opt}}^2 = (MG_{\text{opt}} - V)^T(MG_{\text{opt}} - V)$$

[4] Terhardt 2000 gives a stretch of 2% over 3 octaves as typical for a piano. If that were a uniform stretch of the scale (which it isn't, but never mind) it would be $\log(2^3 \times 1.02)/\log(2^3) = 1.010$, so about a 1% stretch the way I measure it. While you can explain that by the inharmonicity of piano strings, similar stretches have been observed with truly harmonic timbres. The optimal *shrinkage* of the 12 note equally tempered scale by 0.1% or so certainly can't explain it.

[5] Equation 28 is well known. See, for example, Kolman & Hill 2003, p. 334.

Table 1: Tenney-weighted prime errors (cent/oct) for pure octaves, optimal 5-limit errors (cent/oct) and scale stretches (%) for various equal temperaments

	Tenney-weighted prime errors					TOP-RMS		TOP-Max	
Mapping	3:1	5:1	RMS	Max	STD	Stretch	Error	Stretch	Error
$5, 8, 12$	11.39	40.35	24.20	40.35	16.99	-1.44	16.74	-1.65	19.84
$7, 11, 16$	-10.25	-18.72	12.32	18.72	7.65	0.81	7.71	0.79	9.43
$8, 13, 19$	30.31	27.43	23.60	30.31	13.66	-1.59	13.44	-1.25	14.97
$9, 14, 21$	-22.26	5.89	13.30	22.26	12.13	0.45	12.18	0.69	14.18
$10, 16, 23$	11.39	-11.33	9.27	11.39	9.27	-0.01	9.27	-0.00	11.36
$11, 17, 25$	-29.91	-25.43	22.67	29.91	13.17	1.55	13.38	1.26	15.14
$12, 19, 28$	-1.23	5.89	3.48	5.89	3.11	-0.13	3.11	-0.19	3.56
$13, 21, 30$	23.03	-7.36	13.96	23.03	12.95	-0.45	12.89	-0.65	15.10
$15, 24, 35$	11.39	5.89	7.40	11.39	4.65	-0.48	4.63	-0.47	5.67
$16, 25, 37$	-17.01	-4.87	10.21	17.01	7.15	0.61	7.19	0.71	8.56
$17, 27, 40$	2.48	16.03	9.36	16.03	7.04	-0.51	7.01	-0.66	7.96
$18, 29, 42$	19.80	5.89	11.93	19.80	8.30	-0.71	8.24	-0.82	9.82
$19, 30, 44$	-4.55	-3.17	3.20	4.55	1.91	0.21	1.91	0.19	2.28
$20, 32, 47$	11.39	14.51	10.65	14.51	6.23	-0.72	6.19	-0.60	7.21
$21, 33, 49$	-10.25	5.89	6.82	10.25	6.67	0.12	6.68	0.18	8.09
$22, 35, 51$	4.50	-1.94	2.83	4.50	2.70	-0.07	2.70	-0.11	3.22
$23, 36, 53$	-14.95	-9.09	10.10	14.95	6.15	0.67	6.19	0.63	7.52
$25, 40, 58$	11.39	-1.00	6.60	11.39	5.62	-0.29	5.60	-0.43	6.16
$26, 41, 60$	-6.09	-7.36	5.51	7.36	3.21	0.37	3.22	0.31	3.69
$27, 43, 63$	5.78	5.89	4.76	5.89	2.75	-0.32	2.74	-0.24	2.94
$28, 44, 65$	-10.25	-0.26	5.92	10.25	4.77	0.29	4.78	0.43	5.15
$29, 46, 67$	0.94	-5.99	3.50	5.99	3.07	0.14	3.07	0.21	3.47
$31, 49, 72$	-3.27	0.34	1.90	3.27	1.63	0.08	1.63	0.12	1.81

$$
\begin{aligned}
&= G_{\text{opt}}^T M^T M G_{\text{opt}} - 2V^T M G_{\text{opt}} + V^T V \\
&= V^T M (M^T M)^{-1} M^T M G_{\text{opt}} \\
&\quad -2V^T M G_{\text{opt}} + V^T V \\
&= V^T M G_{\text{opt}} - 2V^T M G_{\text{opt}} + V^T V \\
E_{\text{opt}}^2 &= V^T V - V^T M G_{\text{opt}} \qquad (29)
\end{aligned}
$$

This happens to be related to the sum deviation, which is interesting.

The normalized mean squared error follows as

$$
\begin{aligned}
\left\langle E_{\text{opt}}^2 \right\rangle_V &= \frac{V^T V - V^T M G_{\text{opt}}}{V^T V} \\
&= 1 - \frac{V^T M G_{\text{opt}}}{V^T V} \\
\left\langle E_{\text{opt}}^2 \right\rangle_V &= 1 - \left\langle M G_{\text{opt}} \right\rangle_V \qquad (30)
\end{aligned}
$$

Because you subtract two numbers close to 1, for an accurate temperament the result can be lost by the floating point precision.

It happens that you can also write the normalized mean squared error as

$$
\left\langle E_{\text{opt}}^2 \right\rangle_V = \frac{\left| \left\langle (M - V \left\langle M \right\rangle_V)^2 \right\rangle_V \right|}{\left| \left\langle M^2 \right\rangle_V \right|} \qquad (31)
$$

Here, the normalized mean and mean squared are defined as in Equation 21 on page 7 and Equation 14 on page 6. However, because M is not a column vector, they give vectors as output. $|A|$ is the determinant of A (Clapham pp. 68–69). This gives identical results to Equation 30 but the proof is involved, and so in Appendix B.

The numerator of Equation 31 is a generalization of the standard deviation. That means you can also write Equation 31 as

$$
\left\langle E_{\text{opt}}^2 \right\rangle_V = \frac{\left| \left\langle M^2 \right\rangle_V - \left\langle M \right\rangle_V^2 \right|}{\left| \left\langle M^2 \right\rangle_V \right|} \qquad (32)
$$

with $\left\langle A \right\rangle_V^2 = \left\langle A \right\rangle_V^T \left\langle A \right\rangle_V$. You can see that this is similar to Equation 24 on the preceding page (and identical for an equal temperament). The proof that Equations 31 and 32 are equivalent is also in Appendix B.

Table 2: Tenney-weighted prime errors (cent/oct) for pure octaves, optimal 7-limit errors (cent/oct) and scale stretches (%) for various equal temperaments

	Tenney-weighted prime errors						TOP-RMS		TOP-Max	
Mapping	3:1	5:1	7:1	RMS	Max	STD	Stretch	Error	Stretch	Error
$5, 8, 12, 14$	11.39	40.35	−3.14	21.02	40.35	17.16	−1.02	16.98	−1.53	21.41
$6, 10, 14, 17$	61.86	5.89	11.10	31.56	61.86	24.65	−1.66	24.24	−2.51	30.15
$7, 11, 16, 19$	−10.25	−18.72	−39.78	22.57	39.78	14.63	1.44	14.84	1.69	20.23
$8, 13, 19, 23$	30.31	27.43	28.91	25.04	30.31	12.55	−1.78	12.33	−1.25	14.97
$9, 14, 21, 25$	−22.26	5.89	−12.64	13.14	22.26	10.95	0.60	11.02	0.69	14.18
$10, 16, 23, 28$	11.39	−11.33	−3.14	8.18	11.39	8.15	0.06	8.15	−0.00	11.36
$11, 17, 25, 30$	−29.91	−25.43	−34.23	26.04	34.23	13.30	1.89	13.55	1.45	17.36
$12, 19, 28, 34$	−1.23	5.89	11.10	6.32	11.10	4.94	−0.33	4.92	−0.41	6.14
$13, 21, 31, 37$	23.03	32.40	16.58	21.54	32.40	11.82	−1.49	11.64	−1.33	15.98
$14, 22, 32, 39$	−10.25	−18.72	−9.25	11.63	18.72	6.63	0.80	6.68	0.79	9.43
$15, 24, 35, 42$	11.39	5.89	−3.14	6.60	11.39	5.57	−0.30	5.56	−0.34	7.24
$16, 25, 37, 45$	−17.01	−4.87	2.20	8.91	17.01	7.43	0.41	7.46	0.62	9.66
$17, 27, 40, 48$	2.48	16.03	6.91	8.82	16.03	6.11	−0.53	6.08	−0.66	7.96
$18, 29, 42, 51$	19.80	5.89	11.10	11.73	19.80	7.27	−0.76	7.22	−0.82	9.82
$19, 30, 44, 53$	−4.55	−3.17	−7.64	4.72	7.64	2.75	0.32	2.76	0.32	3.83
$20, 32, 47, 57$	11.39	14.51	18.23	12.97	18.23	6.81	−0.91	6.75	−0.75	9.05
$21, 33, 49, 59$	−10.25	5.89	0.93	5.93	10.25	5.87	0.07	5.87	0.18	8.09
$22, 35, 51, 62$	4.50	−1.94	4.63	3.37	4.63	2.85	−0.15	2.85	−0.11	3.28
$23, 36, 53, 64$	−14.95	−9.09	−10.58	10.22	14.95	5.44	0.72	5.48	0.63	7.52
$24, 38, 56, 67$	−1.23	5.89	−6.71	4.51	6.71	4.48	0.04	4.48	0.03	6.30
$25, 40, 58, 70$	11.39	−1.00	−3.14	5.93	11.39	5.64	−0.15	5.63	−0.34	7.24
$26, 41, 60, 73$	−6.09	−7.36	0.14	4.77	7.36	3.43	0.28	3.44	0.30	3.76
$27, 43, 63, 76$	5.78	5.89	3.19	4.42	5.89	2.40	−0.31	2.39	−0.24	2.94
$28, 44, 65, 78$	−10.25	−0.26	−9.25	6.90	10.25	4.82	0.41	4.84	0.43	5.15
$29, 46, 67, 81$	0.94	−5.99	−6.09	4.30	6.09	3.27	0.23	3.28	0.22	3.52
$30, 48, 70, 85$	11.39	5.89	11.10	8.48	11.39	4.64	−0.59	4.62	−0.47	5.67
$31, 49, 72, 87$	−3.27	0.34	−0.39	1.65	3.27	1.43	0.07	1.43	0.12	1.81

2.5 Rank 2 Temperaments

For a rank 2 temperament, the weighted mapping has two columns, M_0 and M_1, which correspond to either the weighted mappings of two equal temperaments or the weighted period and generator mappings. Either way they'll have corresponding generators g_0 and g_1. Equation 27 on page 8 becomes

$$E^2 = (M_0 g_0 + M_1 g_1 - V)^T (M_0 g_0 + M_1 g_1 - V) \quad (33)$$

The optimal generators follow from Equation 28 on page 8

$$\begin{aligned}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}_{\text{opt}} &= \left[\begin{pmatrix} M_0^T \\ M_1^T \end{pmatrix} \begin{pmatrix} M_0 & M_1 \end{pmatrix} \right]^{-1} \begin{pmatrix} M_0^T \\ M_1^T \end{pmatrix} V \\ &= \begin{pmatrix} M_0^T M_0 & M_0^T M_1 \\ M_1^T M_0 & M_1^T M_1 \end{pmatrix}^{-1} \begin{pmatrix} M_0^T V \\ M_1^T V \end{pmatrix} V \\ &= \frac{\begin{pmatrix} M_1^T M_1 & -M_0^T M_1 \\ -M_1^T M_0 & M_0^T M_0 \end{pmatrix} \begin{pmatrix} M_0^T V \\ M_1^T V \end{pmatrix}}{M_0^T M_0 M_1^T M_1 - \left(M_0^T M_1\right)^2}\end{aligned}$$

$$g_0 = \frac{M_1^T M_1 M_0^T V - M_0^T M_1 M_1^T V}{M_0^T M_0 M_1^T M_1 - \left(M_0^T M_1\right)^2} \quad (34)$$

$$g_1 = \frac{M_0^T M_0 M_1^T V - M_0^T M_1 M_0^T V}{M_0^T M_0 M_1^T M_1 - \left(M_0^T M_1\right)^2} \quad (35)$$

For Tenney weighting, they simplify to

$$g_{0opt} = \frac{\langle M_0 \rangle \langle M_1^2 \rangle - \langle M_1 \rangle \langle M_0 M_1 \rangle}{\langle M_0^2 \rangle \langle M_1^2 \rangle - \langle M_0 M_1 \rangle^2} \quad (36)$$

$$g_{1opt} = \frac{\langle M_1 \rangle \langle M_0^2 \rangle - \langle M_0 \rangle \langle M_0 M_1 \rangle}{\langle M_0^2 \rangle \langle M_1^2 \rangle - \langle M_0 M_1 \rangle^2} \quad (37)$$

From Equation 32 on the previous page and the observation that the denominator works out as the de-

nominators of Equations 36 and 37 (which follows from Cramer's rule (Clapham p. 58)) the TOP-RMS error is

$$\left\langle e^2_{\text{opt}} \right\rangle = \frac{\sigma^2_{M_0}\sigma^2_{M_1} - \sigma^2_{M_0 M_1}}{\left\langle M_0^2 \right\rangle \left\langle M_1^2 \right\rangle - \left\langle M_0 M_1 \right\rangle^2} \tag{38}$$

where σ^2_X is the variance of X (Weisstein 2006) and σ_{XY} is the covariance of X and Y (Weisstein 2006a)

$$\sigma^2_X = \left\langle X^2 \right\rangle - \left\langle X \right\rangle^2 \tag{39}$$

$$\sigma_{XY} = \left\langle XY \right\rangle - \left\langle X \right\rangle \left\langle Y \right\rangle \tag{40}$$

2.6 Rank 2 Example

Let's calculate the TOP-RMS tuning and error for 7-limit meantone as an example. The mapping is

$$\left| \begin{array}{cccccc} \langle & 1 & 2 & 4 & 7 & | \\ \langle & 0 & -1 & -4 & -10 & | \end{array} \right\rangle \tag{41}$$

This is in the standard form where the first entry is the period mapping (the mapping by the generator that equally divides the octave) and the second entry is the generator mapping (the mapping by the generator that's independent of the octave). In this case, the period is the octave, and so the top line tells you how many octaves contribute to each prime interval. The generator here is a perfect fourth, so the bottom line tells you where each prime interval comes on the spiral of fourths.

Table 3 on the following page shows the process of calculating the means used in Equations 36 and 37 on the previous page. M_0 and M_1 are calculated by dividing entries of the mapping by the buoyancy (logs to base two, so octaves). The other columns are calculated from M_0 and M_1. If you have the time, I expect you can check the whole table with a pocket calculator.

To get the optimal tempered octave, work through Equation 36 on the preceding page.

$$\begin{aligned} g_{0opt} &= \frac{1.620 \times 4.014 - -1.479 \times -3.161}{2.944 \times 4.014 - (-3.161)^2} \\ &= \frac{1.8276}{1.8253} \\ &= 1.0013 \pm 0.0005\,\text{oct} \\ & \quad 1.0013 \times 1200 = 1201.6 \pm 0.6\,\text{cent} \end{aligned}$$

The correct value (which is in Table 7 on page 27 if you want to check) is 1201.2 cents. So this result is good enough to within the rounding error – and you can check it with a pocket calculator!

Next, get the optimal generator from Equation 37 on the preceding page and noticing that the denominator's the same as Equation 36 on the previous page

$$\begin{aligned} g_{1opt} &= \frac{-1.479 \times 2.944 - 1.62 \times -3.161}{1.8253} \\ &= \frac{0.7666}{1.8253} \\ &= 0.41999 \pm 0.00005\,\text{oct} \\ & \quad 0.41999 \times 1200 = 503.99 \pm 0.06\,\text{cent} \end{aligned}$$

This matches the correct value of 504.03 cents to 4 figure accuracy.

To get the TOP-RMS error, plug these generators into Equation 33 on the preceding page. Four figures in should give nearly four decimal places of octaves out.

$$\begin{aligned} \left\langle e^2 \right\rangle = \frac{1}{4}[&(1 \times 1.0013 + 0 - 1)^2 \\ &+(1.262 \times 1.0013 \\ &\quad -0.631 \times 0.41999 - 1)^2 \\ &+(1.723 \times 1.0013 \\ &\quad -1.723 \times 0.41999 - 1)^2 \\ &+(2.493 \times 1.0013 \\ &\quad -3.562 \times 0.41999 - 1)^2] \end{aligned}$$

$$\begin{aligned} &= \frac{\begin{array}{c} 0.0013^2 + (-0.00014)^2 \\ +0.00160^2 + 0.00024^2 \end{array}}{4} \\ &= \frac{0.000004327}{4} \\ &= 0.000001082 \\ \sqrt{\left\langle e^2 \right\rangle} &= \sqrt{0.000001082} = 0.00104\,\text{oct} \\ & \quad 0.00104 * 1200 = 1.25\,\text{cent/oct} \end{aligned}$$

The correct value, from Table 7, is 1.382. So figures accurate to about a cent give a result accurate to about a cent per octave.

3 Worst Weighted Error

The simplest way to think about the error of a set of intervals is to take the worst case as representative of the whole set. How does this fit with the criteria I gave?

1. In an important special case, we'll see that the worst weighted error for a prime is also the worst weighted error for any interval. So the prime intervals are not an arbitrary choice – we can derive a property of all simple intervals from them.

2. If you only look at the worst error, that tells you nothing about the other ones.

3. If you assume that mistuning gets worse the larger it gets, only looking at the most mistuned interval gives you an idea of how the mistuning of a chord will sound.

Table 3: Intermediate Calculations for TOP-RMS Meantone. Buoyancy in octaves.

Prime	Mapping		Buoyancy	M_0	M_1	M_0^2	M_1^2	M_0M_1
2:1	1	0	1.000	1.000	0.000	1.000	0.000	0.000
3:1	2	−1	1.585	1.262	−0.631	1.592	0.398	−0.796
5:1	4	−4	2.322	1.723	−1.723	2.968	2.968	−2.968
7:1	7	−10	2.807	2.493	−3.562	6.217	12.688	−8.882
			Sums:	6.478	−5.916	11.777	16.054	−12.646
			Means:	1.620	−1.479	2.944	4.014	−3.161

4. The worst error might be stupidly large, but at least you know you have a stupidly large error and so anything else calculated from all the primes is irrelevant.

The worst error is useful for telling you if any errors fall into your "moderate" or "large" categories. You may want to use the worst error as a barrier to ensure you only look at acceptable temperaments, and then use another average to compare those temperaments with each other. Unweighted errors of a deliberately chosen, finite set of intervals may tell you more about which category the errors fall in.

The worst error, from Equation 6 on page 3 becomes

$$\max(e) = \max(|w - v|) \tag{42}$$

You can also write it as

$$\max(e) = \max[\max(w - v), -\min(w - v)] \tag{43}$$

because the maximum weighted error always has to be either the largest weighted deviation, or the negative deviation with the largest magnitude.

For Tenney weighting, each v_i is 1. In that case

$$\begin{aligned}\max(e) &= \max[\max(w - 1), -\min(w - 1)] \\ &= \max\begin{bmatrix}\max(w) - 1, \\ -\min(w) + 1\end{bmatrix} \\ &= \max\begin{bmatrix}\max(w) - 1, \\ 1 - \min(w)\end{bmatrix}\end{aligned} \tag{44}$$

Now, try stretching the tempered scale by a uniform amount α. The relative sizes of the weighted primes aren't affected by the scale stretch. So $\max(\alpha w) = \alpha \max(w)$ and $\min(\alpha w) = \alpha \min(w)$. The worst error is now

$$\begin{aligned}\max(e) &= \max\begin{bmatrix}\max(\alpha w) - 1, \\ 1 - \min(\alpha w)\end{bmatrix} \\ &= \max\begin{bmatrix}\alpha \max(w) - 1, \\ 1 - \alpha \min(w)\end{bmatrix}\end{aligned} \tag{45}$$

This means the worst error only depends on the same two primes whatever the scale stretch. We need to find a value for α that balances the errors in each. The smaller α gets, the smaller $\alpha \max(w) - 1$ gets, and the larger $1 - \alpha \min(w)$ gets. The larger α gets, the larger $\alpha \max(w) - 1$ gets, and the smaller $1 - \alpha \min(w)$ gets. So as long as $\alpha \max(w) - 1$ is the maximum error, we make α smaller, and as long as $1 - \alpha \min(w)$ is the maximum error, we make α larger. The smallest max-error (the minimax error) will occur when

$$\begin{aligned}\alpha \max(w) - 1 &= 1 - \alpha \min(w) \\ \alpha[\max(w) + \min(w)] &= 1 + 1\end{aligned}$$

$$\alpha = \frac{2}{\max(w) + \min(w)} \tag{46}$$

This tells you the scale stretch for the TOP-max tuning.

The TOP-max stretch isn't much different to the TOP-RMS stretch. The difference between stretching and not stretching is more important than the differences between different stretches.

If you substitute Equation 46 into the first part of Equation 45, you get[6]

$$\begin{aligned}\max(e_{\text{opt}}) &= \frac{2}{\max(w) + \min(w)} \max(w) - 1 \\ &= \frac{2 \max(w)}{\max(w) + \min(w)} \\ &\quad - \frac{\max(w) + \min(w)}{\max(w) + \min(w)} \\ \max(e_{\text{opt}}) &= \frac{\max(w) - \min(w)}{\max(w) + \min(w)}\end{aligned} \tag{47}$$

3.1 Higher Rank Temperaments

Unfortunately, I don't have a formula for finding the TOP-max error for an arbitrary regular temperament.

[6] Equations 46 and 47 were originally given by Erlich 2004.

For the practical calculation I use a linear programming library. It's much slower than the least-squares optimization library that does the RMS calculation. There is a simple method for tempering out a single comma, but the general case is difficult because it isn't continuously differentiable. Anyway, if you're tempering out a ratio n:d, the TOP-max error is given by Erlich 2006

$$\max(e) = \frac{\log\left(\frac{n}{d}\right)}{\log(nd)} \tag{48}$$

which is the same as the Tenney-weighted error of n:d by Equation 8 on page 4.

One interesting property of the TOP-max error is that it's also the highest weighted error for any interval within the prime limit, although you only need to consider the primes to calculate it Erlich 2006. Provided you think Tenney-weighted error is a good measure of mistuning, the worst such error tells you the worst mistuning of *any* interval in the prime limit.

As the TOP-RMS error is more efficient to calculate for many important cases, it's useful to know that it's guaranteed to be no larger than the TOP-max error. If you're searching through a large number of regular temperaments to find those with a TOP-max error smaller than a given value, you can first calculate the TOP-RMS error and if it's already to large you can save the trouble of calculating the TOP-max error.

4 Octave Independence

Equation 25 on page 8 and Equation 47 on the preceding page both give a TOP error that's independent of the scale stretch. You can verify this by replacing w with αw and noticing that it doesn't alter the result. That means it's possible to calculate the TOP error without optimizing the octaves! So, TOP errors are a simple yet correct measure even when the octaves are pure.

For higher rank temperaments, you need to find the optimal generators, and then unstretch the scale so that octaves are pure. Where the period is g_0 octaves, and there are M_{00} periods to an octave, the size of the octave is $M_{00}g_0$. For the optimal unstretched rank 2 TOP-RMS, you divide Equation 37 on page 10 by the size of an octave given by Equation 36 on page 10 to get

$$g_{\text{opt}} = \frac{1}{M_{00}} \frac{\langle M_1 \rangle \langle M_0^2 \rangle - \langle M_0 \rangle \langle M_0 M_1 \rangle}{\langle M_0 \rangle \langle M_1^2 \rangle - \langle M_1 \rangle \langle M_0 M_1 \rangle} \tag{49}$$

4.1 Approximations

Equation 25 on page 8 and Equation 47 on the preceding page are of the same form: a kind of deviation divided by a kind of average. For the TOP-max error, the denominator is proportional to the optimal scale stretch (see Equation 46 on the previous page). Because the scale stretch is usually small, the denominators don't affect the result much.

Note that the optimally stretched prime errors (the middle bars of Figures 3 and 4 on page 7) are almost identical to the unstretched errors measured relative to the average error, rather than the optimal tuning for each interval (the right hand bars). This shows you that the deviations don't change much with the stretch: only the absolute values.

A 1% scale stretch will only alter the TOP error by about 1%, and this means octaves will be sharp or flat by 12 cents when you optimize the stretch. That's no problem given that 12 cents is quite a hefty scale stretch and a 1% error is negligible given that we aren't measuring anything real here. The denominator for the TOP-RMS case is the RMS of the weighted primes for the unstretched scale. Tables 1 on page 9 and 2 on page 10 show that the RMS Tenney-weighted error for an unstretched temperament is also within 12 cents for a good temperament. I don't think an ordering of tables by either TOP error would be affected by only taking the deviation parts. That gives us:

$$\sqrt{\langle e_{\text{opt}}^2 \rangle} \simeq \sqrt{\langle w^2 \rangle - \langle w \rangle^2} \tag{50}$$

$$\max(e) \simeq \frac{\max(w) - \min(w)}{2} \tag{51}$$

They can also be written in terms of weighted deviations, E_i where $E_i = w_i - 1$. Hopefully you'll see it as obvious that Equation 51 is the same as

$$\max(e) \simeq \frac{\max(E) - \min(E)}{2} \tag{52}$$

For Equation 50 either try substituting in $w_i = E_i + 1$ or consult a statistics text such as Wetherill 1972 (p. 29) to get

$$\sqrt{\langle e_{\text{opt}}^2 \rangle} \simeq \sqrt{\langle E^2 \rangle - \langle E \rangle^2} \tag{53}$$

Equation 50 is given in the "STD" columns of Tables 1 on page 9 and 2 on page 10. You can see that, as expected, it doesn't differ from the TOP-RMS error by very much. The smaller the errors get, the closer the STD approximation is, so that in some cases they're the same to three figure accuracy.

One thing to note about these approximations is that they don't only *work* with unoptimized octaves, they

depend upon the octaves not being optimized! Both deviations of errors will get smaller the more you shrink the scale, so that if you shrink everything to a unison the total errors become zero. So remember not to try optimizing them for scale stretch, or you'll get infinities coming out.

Equation 51 on the preceding page gives the true TOP-max error if you apply the TOP scale stretch. That's because the denominator tells you how much to stretch the scale to reach the optimum. When you're already at the optimum point, no additional stretch is required.

Equation 50 on the previous page is nice for a rank 2 temperament, because it means the sum squared error is a quadratic function of the generator. You can either think of the period as being fixed or the whole thing defined in projective space and the generator representing the ratio of the stretched generator to the stretched octave. The weighted primes are then written as

$$w_i = \frac{M_{i0}}{M_{00}} + M_{i1}g \tag{54}$$

with M_{00} as the number of periods to an octave and g as the generator (previously written as g_1). Substituting this into Equation 50 on the preceding page gives

$$\begin{aligned} e^2 &= \left\langle \left(\frac{M_0}{M_{00}} + M_1 g \right)^2 \right\rangle - \left\langle \frac{M_0}{M_{00}} + M_1 g \right\rangle^2 \\ &= g^2 \left(\left\langle M_1^2 \right\rangle - \left\langle M_1 \right\rangle^2 \right) \\ &\quad + \frac{2g}{M_{00}} \left(\left\langle M_0 M_1 \right\rangle - \left\langle M_0 \right\rangle \left\langle M_1 \right\rangle \right) \\ &\quad + \frac{\left\langle M_0^2 \right\rangle - \left\langle M_0 \right\rangle^2}{M_{00}^2} \\ &= \sigma_{M_1}^2 g^2 + \frac{2\sigma_{M_0 M_1}}{M_{00}} g + \frac{\sigma_{M_0}^2}{M_{00}^2} \\ e^2 &= \sigma_{M_1}^2 (g - g_{\text{opt}})^2 + e_{\text{opt}}^2 \end{aligned} \tag{55}$$

This is a standard quadratic equation in g. You can work out the value of g_{opt} that gives the right coefficient on g in the expanded equation.

$$\begin{aligned} g_{\text{opt}} &= \frac{1}{M_{00}} \frac{\left\langle M_0 \right\rangle \left\langle M_1 \right\rangle - \left\langle M_0 M_1 \right\rangle}{\left\langle M_1^2 \right\rangle - \left\langle M_1 \right\rangle^2} \\ g_{\text{opt}} &= -\frac{\sigma_{M_0 M_1}}{M_{00} \sigma_{M_1}^2} \end{aligned} \tag{56}$$

The variance and covariance are as in Equations 39 and 40 on page 11 [7].

[7] If you want formulas like this to work with general weights, you can replace means with normalized means as in Equations 14 on page 6 and 21 on page 7 to get a weighted variance of

$$\sigma_X^2 \to \left\langle X^2 \right\rangle_V - \left\langle X \right\rangle_V^2 \tag{57}$$

The weighted covariance would need another special-case symbol, so lets write it in matrix form:

$$\sigma_{XY} \to \frac{X^T Y}{V^T V} - \frac{V^T X Y^T V}{V^T V V^T V} \tag{58}$$

Substituting this optimal generator into Equation 55 gives

$$\begin{aligned} e_{\text{opt}}^2 &= \frac{\sigma_{M_0}^2}{M_{00}^2} - g_{\text{opt}}^2 \sigma_{M_1}^2 \\ e_{\text{opt}}^2 &= \frac{\sigma_{M_0}^2 \sigma_{M_1}^2 - \sigma_{M_0 M_1}^2}{M_{00}^2 \, \sigma_{M_1}^2} \end{aligned} \tag{59}$$

Equation 51 on the previous page is also nice to work with, because it makes the error a piecewise linear function of the generator size, which I hope has a single well-defined minimum. I know an algorithm for finding the minimum of such a function. What you do is choose two values that you expect to lie either side of the minimum. Then, you find the point where their two gradients meet. You then replace one of the points with this new one, such that the minimum must lie between them. Before long, you'll hit the exact minimum point. It would be nice if that is also a stretch of the TOP-max tuning, but I can't prove it.

You can even make a case for Equations 50 and 51 on the preceding page being the natural TOP-like errors for untempered octaves without taking them as approximations of error measures that do temper the octaves. They fix the main problem with calculating an average of the prime errors without optimizing the octaves: that large intervals have the same weight as small intervals of a given complexity, but small intervals are more likely to be harmonically obtrusive.

One way to shift the bias towards small intervals is to consider the intervals between the prime intervals, instead of the prime intervals themselves. That's what Equation 51 on the previous page does. It averages the lowest and highest weighted deviations in the prime intervals to give the highest weighted error in an interval between prime intervals. But that's only one more approximation on top of all the other approximations we're dealing with here.

Even though octaves are kept pure, remember to consider them when calculating the means. If you use vectors that don't include the octave component then you'll have to add the octave terms manually. If octaves are pure, they'll always have zero error, and so the octave component of w is 1. The octave component of M_0 tells you how many periods there are to an

octave. The octave component of M_1 is always zero. All this assuming weights and intervals are in terms of octaves, which is assumed for a lot of these equations.

4.2 Paired Equal Temperaments

Sometimes it's useful to know the optimal error for a regular temperament class given two equal temperaments that arise as special-case tunings. The optimal standard deviation of the weighted primes, as in Equation 50 on page 13, as a function of the errors E_1 and E_2 of the defining equal temperaments, is

$$e_{\text{opt}}^2 = \frac{\sigma_{E_0}^2 \sigma_{E_1}^2 - \sigma_{E_0 E_1}^2}{\sigma_{E_0 - E_1}^2} \tag{60}$$

This is good because the errors tend to be small, and a calculation that only involves differences between small numbers doesn't suffer as much from rounding error as one that involves small differences between larger numbers. Showing it is a bit difficult so I left it to Appendix C.

It's nice that the denominator becomes zero if you feed the same ET in twice, because the resulting temperament would make no sense.

4.3 Kees Weighting

The Kees metric is a more systematic way of dealing with octave-equivalent errors [8]. This is like a Tenney metric, but with two differences:

- Factors of 2 are ignored.
- For an interval $n{:}d$, the overall complexity is the highest complexity of n and d rather than their sum.

Following these rules, we can define the weighted error for an interval x in a similar form to Equation 8 on page 4

$$e(x) = \frac{\left|\sum_i x_i d_i\right|}{\max\left[\sum_i \max(x_i b_i, 0), \sum_i \max(-x_i b_i, 0)\right]} \tag{61}$$

Either you only consider odd primes ($i > 0$) or set the buoyancy of octaves to zero ($b_0 = 0$). The deviation of octaves will always be zero ($d_0 = 0$) because the Kees metric is only valid for tunings with pure octaves. Intervals that only consist of octaves will give a zero-division error. Don't worry too much because unisons do the same thing with a Tenney metric. In practice they should have zero weighted error.

The logic behind the Kees metric is that we want the harmonic distance of an interval to be the same if we make it larger by any number of octaves. Because small intervals tend to be the most harmonically obtrusive, we score each interval according to its smallest possible manifestation. That means taking the smaller odd part of the ratio, and adding factors of two to it until it gets to be the same size as the other part. For example, the smaller odd part of 9:4 is 9. So we beef up the 4 to be about the same size, to make it 8. That gives us 9:8, so 9:4 is weighted as if it were 9:8. The Tenney harmonic distance would then be the log of 9×8.

For the Kees metric, we cheat and add a fractional number of octaves so that the numerator and denominator are of equal length. That means instead of 9:4 turning into 9:8, it becomes 9:9. The Tenney harmonic distance would be the logarithm of 9×9, or twice the logarithm of 9. The Kees harmonic distance is half of the Tenney harmonic distance of this theoretical, infinitely small interval.

With Tenney weighted errors, we know that the worst weighted error of the prime intervals is the same as the worst weighted error in any interval in the prime limit. With Kees weighted errors, this is no longer the case. For example, you probably know that in 12 note equal temperament the minor third has a greater deviation from just intonation than either the major third or perfect fifth. With Kees weighting, perfect fifths and major thirds have the same weight as 3:1 and 5:4 respectively, and therefore the same weighted errors. However a minor third, interpreted as 6:5 has a Kees buoyancy of the logarithm of 5, and so the same weighting as a major third. Because fifths are very well tuned, their weighted error is small. So the minor third has a higher weighted error than either the major third or perfect fifth.

It happens that the worst Kees error for a regular temperament is double the complexity in Equation 52 on page 13. So, the worst Kees error for a regular temperament is the same as the approximate TOP-max error with pure octaves, normalized so that Kees and Tenney errors are comparable. Proving this is fairly difficult so I moved it to Appendix D.

[8] From a proposal of Kees van Prooijen. See Smith 2006a, although it's really about something else.

5 Complexity and Badness

5.1 Criteria for a Complexity Measure

What is the complexity of a temperament class? Simply put, it's the number of notes you need. But that depends on what you want to do. Some desirable properties of complexity are:

1. The more complex simple intervals become, the more complex the temperament.

2. All tunings of the same temperament class should have the same complexity.

3. All mapping matrices for the same temperament class should have the same complexity.

4. It should generalize to any regular temperament class.

Any temperament class will have some intervals that are a small number of generators. That's no use if they're not intervals you're likely to want to use. For a simple temperament class, then, simple ratios from JI should map to intervals with a small number of generators. The problem then becomes how to count generators, and how to balance different intervals.

A given temperament class should have the same complexity regardless of how it's tuned. Different tunings will be written down the same way and map to keyboards the same way. Sometimes complexity counts the number of notes to an octave. That appears to break this rule, but you can always call an octave the interval 2:1 maps to rather than always being 1200 cents (except for the cases where 2:1 is outside the temperament, which is where the definitions get tricky). As all temperaments in a class share the same mapping, it's natural that the complexity should be a function of that mapping.

There are always different ways of writing the mapping for a temperament. The complexity shouldn't depend on the representation. So all choices of generator for the same temperament class should have the same complexity.

Measuring the complexity is easy for equal temperaments. All you do is count the number of notes to the octave. For rank 2 temperaments you count the number of octave-equivalent generators in some way. It gets harder when you consider higher rank temperaments. Although special-case measures have their uses, we really want something that works for any rank and any number of primes.

5.2 Max Weighted Complexity

The complexity of an interval tells you how many notes you need in your scale to get that interval to be included. Take a scale made up as a chain of generators. The same note is repeated in each octave. If the complexity of an interval is larger than the number of notes in the scale, then the interval isn't in the scale. Otherwise, the number of intervals of that kind you get in the scale is the difference between the number of notes in the scale and the complexity of the interval.

The tuning of an interval, x, in a rank 2 temperament with pure octaves is given by

$$t(x) = \frac{m_0 \cdot x}{m_{00}} + m_1 \cdot xg \tag{62}$$

where m_{00} is the number of periods to an octave, m_0 is the unweighted period mapping, m_1 is the unweighted generator mapping, $m_0 \cdot x$ is the dot product of m_0 and x taken as vectors, and g is the size of the generator. The complexity of x is

$$k(x) = m_{00}|m_1 \cdot x| \tag{63}$$

When we use odd limits, the complexity of a rank 2 temperament as a whole is the highest complexity of all the intervals in the relevant odd limit. For prime limits, we have to define a weighted complexity. With post-weighting, that's

$$k(x) = m_{00}\frac{\left|\sum_i m_{i1}x_i\right|}{\sum_i b_i|x_i|} \tag{64}$$

The trouble with this equation is that adding an octave to an interval reduces its complexity. Really, the complexity should be octave-equivalent. For octave-equivalent weighting that's like Tenney weighting, we can use the Kees metric. For an interval between primes i and j, the Kees-weighted complexity is

$$k_{ij} = m_{00}\frac{|m_{i1} - m_{j1}|}{\max(b_i, b_j)} \tag{65}$$

which can't be higher than

$$k_{ij} = M_{00}[|M_{i1}| + |M_{j1}|] \tag{66}$$

where M_1 is the weighted generator mapping and M_{00} is still the number of periods to an octave. Either M_{i1} or M_{j1} is underweighted, because the unweighted mappings should have both been divided by the higher buoyancy of the two. Because it was divided by too small a buoyancy, it's a bit bigger than it should be.

But that makes the sum too large, so the real complexity can't be larger.

The complexity of the temperament can be set as the highest Kees complexity of an interval between prime intervals[9].

$$k(M) = M_{00}[\max(M_1) - \min(M_1)] \tag{67}$$

Because the first element of the generator mapping (M_{01}) is always zero, no prime intervals or intervals between prime intervals can have a weighted complexity larger than this.

This complexity formula is only valid for rank 2 temperaments because it involves the mapping of a single period-equivalent generator.

5.3 STD Weighted Complexity

There's also a complexity which derives from the standard deviation (Weisstein 2006) of the weighted mapping, and so is related to the TOP-RMS error.

$$k(M) = M_{00}\sqrt{\langle M_1^2\rangle - \langle M_1\rangle^2} \tag{68}$$

Or, using σ_x for the standard deviation of x,

$$k(M) = M_{00}\sigma_{M_1} \tag{69}$$

Alternatively, following Equation 60 on page 15

$$k(M) = \sigma_{M_0 - M_1} \tag{70}$$

I can't make an obvious case for why a standard deviation should work as complexity. However, think about the complexity of an interval as being the number of generators that make it up. This tells you how fast the tuning of that interval changes as the tuning of the generator changes. The complexity of the temperament as a whole in a given odd-limit is the highest value of the complexity for all intervals within that odd limit. If you plot a function of odd limit error against the generator tuning, the complexity is the highest absolute value of the gradient of this function, multiplied by the number of periods to the octave so that you get a fair comparison between temperaments. The standard deviation does the same thing for the TOP error.

To show this, take Equation 55 on page 14 which shows an approximation to the TOP mean squared error as a simple quadratic equation, and use it to find the gradient of the RMS error.

$$\left\langle e^2\right\rangle = \sigma_{M_1}^2(g - g_{\text{opt}})^2 + \left\langle e_{\text{opt}}^2\right\rangle$$

$$\frac{\mathrm{d}\left\langle e^2\right\rangle}{\mathrm{d}g} = 2g\sigma_{M_1}^2 + \text{const}$$

$$\frac{\mathrm{d}\left\langle e^2\right\rangle^{\frac{1}{2}}}{\mathrm{d}g} = \frac{1}{2}\left\langle e^2\right\rangle^{-\frac{1}{2}}\frac{\mathrm{d}\left\langle e^2\right\rangle}{\mathrm{d}g}$$

$$\frac{\mathrm{d}\sqrt{\left\langle e^2\right\rangle}}{\mathrm{d}g} = \frac{g\sigma_{M_1}^2 + \text{const}}{\sqrt{g^2\sigma_{M_1}^2 + O(g)}}$$

$$\lim_{g\to\infty}\frac{\mathrm{d}\sqrt{\left\langle e^2\right\rangle}}{\mathrm{d}g} = \sigma_{M_1} \tag{71}$$

Equation 55 on page 14 shows the TOP mean squared error as a quadratic function of the generator g. The quadratic curve has a minimum point, which is the optimum tuning. So, the maximum gradient comes when the generator approaches infinity. As Equation 71 shows, this is what we need for the gradient times the number of periods per octave to be what we want for Equation 69 to be a weighted complexity measure.

Note that for the weighted complexity of Equation 69 to be comparable to the other form, you should take half the size of Equation 67 to give

$$k(M) = M_{00}\frac{\max(M_1) - \min(M_1)}{2} \tag{72}$$

Like the max weighted complexity, STD complexity is only valid for rank 2 temperaments.

5.4 Badness

When I search through huge lists of temperaments to find interesting ones, I score them by something called *badness*. That's chosen so that it increases as a temperament gets more complex or further from just intonation.[10] Badness is some function of error and complexity and, although error×complexity weights the error a bit low, it will work.[11]

For an equal temperament, the octave-equivalent RMS error is the standard deviation of the weighted errors, or σ_E. To get an error×complexity badness, multiply by the number of notes to the octave to get $M_0\sigma_E$ or σ_M.

From the error in Equation 59 on page 14 and the complexity in Equation 69, you can derive a rank 2

[9] Gene says this is the formula for max Kees complexity as well, but if he gave a proof I didn't understand it.

[10] Gene defines "badness" as "a function of complexity and error of the temperament, which increases if you increase either complexity or error." (Smith 2004)

[11] Gene prefers logflat badness, defined as

$$ek^{\frac{d}{d-r}} \tag{73}$$

where e is error, k is complexity, d is the number of prime intervals, and r is the rank of the temperament. (*ibid*)

badness of

$$B^2(M_0, M_1) = \sigma^2_{M_0}\sigma^2_{M_1} - \sigma^2_{M_0 M_1} \tag{74}$$

that's simpler to calculate than the error. This is a useful pragmatic choice, and works if M_0 and M_1 represent either a pair of equal temperaments or the generator and period mappings.

5.5 Scalar Complexity

If you compare Equation 74 with Equation 38 on page 11, you can see that the top term of the error is a badness squared, which means that the bottom must be a complexity squared. That gives us a new complexity formula.

$$k(M) = \sqrt{\langle M_0^2\rangle\langle M_1^2\rangle - \langle M_0 M_1\rangle^2} \tag{75}$$

This can be generalized to any rank:

$$k(M) = \sqrt{|\langle M^2\rangle_V|} \tag{76}$$

where $|A|$ is the determinant of A. It's always very close to the STD complexity for the same reason that the TOP-RMS error is always very close to its STD approximation. I call it scalar complexity for reasons that needn't concern us now. Note that it's equal to the denominator of Equations 31 and 32 on page 9:

For equal temperaments it gives the intuitively correct result that the complexity is approximately the number of notes to the octave. So it's measuring the complexity of intervals in some sensible way. It also fulfills the other criteria I stated for a complexity measure. It only depends on the mapping, and doesn't treat octaves as a special case in any way, so it's independent of the tuning. It works for any rank.

To prove that all mappings of a temperament class give the same scalar complexity, note that a mapping M' of the same temperament class as M can always be written as $M' = MA$ where A is a matrix with $|A| = \pm 1$. Then,

$$\begin{aligned} k^2(M') &= \left|\langle M'^2\rangle_V\right| \\ &= \left|\frac{M'^T M'}{V^T V}\right| \\ &= \left|\frac{(MA)^T MA}{V^T V}\right| \\ &= \left|\frac{A^T M^T MA}{V^T V}\right| \end{aligned}$$

From standard properties of determinants[12], it follows that

$$\begin{aligned} k^2(M') &= \left|A^T\right|\left|\frac{M^T M}{V^T V}\right||A| \\ &= (\pm 1)^2\left|\frac{M^T M}{V^T V}\right| \\ &= \left|\langle M^2\rangle_V\right| \end{aligned}$$

Another way of defining scalar complexity for Tenney weighting is with a matrix of averages

$$K_{ij} = \langle M_i M_j\rangle \tag{77}$$

Then, simply

$$k(K) = \sqrt{|K|} \tag{78}$$

5.6 Scalar Badness

If you use TOP-RMS error and scalar complexity (or optimal STD error and STD complexity), the simple badness comes out as the numerators of Equations 31 on page 9 and 32 on page 9. That is,

$$B^2(M) = \left|\left\langle (M - V\langle M\rangle_V)^2\right\rangle_V\right| \tag{79}$$

or

$$B^2(M) = \left|\langle M^2\rangle_V - \langle M\rangle_V^2\right| \tag{80}$$

This is a simple way of measuring badness, but it tends to favor complex but very accurate temperaments. One way of shifting the focus towards simpler temperaments is to add a parameter ϵ so that

$$B^2(M, \epsilon) = \left|\left\langle (M - (1-\epsilon)V\langle M\rangle_V)^2\right\rangle_V\right| \tag{81}$$

where $0 \le \epsilon \le 1$. When ϵ is 0, this is identical to Equation 79. When ϵ is 1, it's identical to scalar complexity as in Equation 76. For values of ϵ close to but not equal to 0, this seems to be a sensible badness measure that favors temperament classes of a certain size. The larger ϵ, the simpler the favored temperaments.

This badness can also be written as.

$$B^2(M, \epsilon) = \left|\langle M^2\rangle_V - (1-\epsilon^2)\langle M\rangle_V^2\right| \tag{82}$$

It's quite complicated to prove so I moved that to Appendix E.

The matrix in the determinant of either formulation is such that the ith row and jth column depends only on the ith and jth columns of M.

$$\begin{aligned} B^2(M, \epsilon) &= \left|\langle M^2\rangle_V - (1-\epsilon^2)\langle M\rangle_V^2\right| \\ &= \left|\frac{M^T M}{V^T V} - (1-\epsilon^2)\frac{M^T V V^T M}{V^T V V^T V}\right| \\ &= \left|M^T\left[\frac{I}{V^T V} - (1-\epsilon^2)\frac{VV^T}{(V^T V)^2}\right]M\right| \\ &= \left|M^T A(V, \epsilon) M\right| \end{aligned} \tag{83}$$

[12] That $|AB| = |A||B|$ (Clapham 1996, p. 69), and $\left|A^T\right| = |A|$, which follows from being able to swap rows and columns without affecting the determinant (*ibid*).

where

$$A(V,\epsilon) = \frac{I}{V^T V} - (1-\epsilon^2)\frac{VV^T}{(V^T V)^2} \tag{84}$$

This means

$$[M^T A(V,\epsilon) M]_{ij} = M_j^T A(V,\epsilon) M_i \tag{85}$$

Naturally, the same principle works for scalar complexity, where $\epsilon = 1$.

For Tenney weighting, the scalar badness ($\epsilon = 0$) of a rank 2 temperament is the covariance (Equation 40 on page 11) of the weighted mappings M_i and M_j.

$$\begin{aligned}\left(\left\langle M^2\right\rangle_V - \left\langle M\right\rangle_V^2\right)_{ij} &= \left\langle M_i M_j\right\rangle - \left\langle M_i\right\rangle \left\langle M_j\right\rangle \\ &= \sigma_{M_i M_j} \qquad (86)\\ & \qquad (87)\end{aligned}$$

That means the general form of the scalar badness for Tenney weighting and $\epsilon = 0$ is the determinant of the covariance matrix (Weisstein 2007).

5.7 Units

Because Tenney weighting leaves so many quantities dimensionless, it's not clear what units complexity is measured in. For now, I've given up on deciding what they are and put "Tenney Weighted" in the tables.

The best way to find the natural units for complexity is using scalar complexity (Equation 76 on the previous page). For Tenney weighting, this is the square root of a determinant of squares of weighted mappings. The squaring and square rooting cancel out, so the units depend on the number of multiplications you do in finding the determinant. That's the rank of the matrix, which is the same as the rank of the temperament class. The complexity is therefore in units of weighted mappings to the power of the rank.

The unweighted generator mapping is simply a list of integers. So the weighted mapping has units of generators per octave. The complexity is therefore $\mathrm{gen}^r\mathrm{oct}^{-r}$ for a rank r temperament.

If the equivalence interval isn't an octave, you can either state the complexity in terms of that equivalence interval, or scale it to be in terms of 2:1 octaves.[13] Which you do partly depends on how seriously you take the equivalence interval. If you really think that the 3:1 equivalence of the Bohlen-Pierce scale takes the place of the octave, then the number of notes to a 3:1 in a given scale is comparable to the number of notes to a 2:1 in a conventional scale. However, if you expect your tunes and orchestration to cover a given acoustic range, you want to know how many notes you're likely to need to cover that range. In that case, the complexity over a standard interval such as the octave is more appropriate.

Note that you can re-write Equation 70 on page 17 to get complexity in the form

$$k = M_{00} M_{01} \sigma_{E_0 - E_1} \tag{88}$$

So, the complexity is the product of the numbers of steps to the octave for each equal temperament and something with dimensions of error. Well, what happens if we replace that error with the optimal error for the temperament? It leaves two mystery numbers that look like the numbers of steps to octaves for equal temperaments. You can rearrange it to get the value for the geometric mean of the mystery numbers.

$$m = \sqrt{\frac{k}{e_{\mathrm{opt}}}} \tag{89}$$

So we have a number that depends on the complexity and error of the temperament, and looks like the number of steps to an equal temperament. What does it mean?

It looks like an alternative complexity measure. For example, this value for optimal 7-limit meantone (12&19) is 34.3. For 7-limit magic (19&22), it's 44.9. For 7-limit miracle (31&41), it's 75.5. For 11-limit miracle, it's 83.1. For 13-limit mystery (29&58), it's 106.4. In each case, the number's a bit bigger than the number of notes to an equal temperament that gets the tuning almost right (31 for meantone, 41 for magic, 72 for miracle, and 87 for mystery). Perhaps it's telling us that if you're using that many notes you may as well use the equal temperament instead.

Of course, it's a peculiar complexity because it depends on the tuning. But it means you can find out how many notes it's worth looking at for whatever tuning you prefer. Think of it as natural size rather than complexity. It means that the schismatic nanotemperament (53&118) becomes bigger in the 7-limit (365.0) than the 5-limit (194.2).

6 Exterior Algebra Applications

6.1 Background

Gene Ward Smith has identified exterior or Grassman algebra as relevant to regular temperaments (Smith 2006). I'll give a brief introduction here, but it likely

[13] This appears to break the rule that complexity shouldn't depend on the tuning. You can get round that by fixing it to the size of the equivalence interval in just intonation.

won't make sense if you start with no knowledge. For more details, see Browne 2001. It isn't complete but it covers the basics. Alternatively, you can skip this section if you find it too difficult.

Exterior algebra involves two operations: an exterior or wedge product and a complement.[14]

The wedge product of two simple elements is always antisymmetric (Brown 2001, `Introduction` p. 5):

$$x \wedge y = -y \wedge x \tag{90}$$

and it follows that the wedge product of an element with itself is zero.

$$x \wedge x = 0 \tag{91}$$

The elements here could be (weighted) linear mappings, intervals in vector form, or the results of wedge products. Geometrically speaking, the wedge product of two vectors is zero if the vectors are parallel, and so the wedge product is the parallel part of the product of vectors.[15]

Each regular temperament class can be uniquely represented by the wedge product of its mappings.[16] All important properties of the temperament class can be derived from this wedge product. A certain vectorization of this wedge product is called the "wedgie" by Gene (Monzo et. al. 2006).

Here, we'll use a weighted wedge product. For equal temperaments, it's simply the weighted mapping. For rank 2 temperaments, it's the product of the equal mappings making up the weighted mapping matrix (these could be the weighted mappings by period and generator, or the weighted mappings or two equal temperaments.)

$$T = M_0 \wedge M_1 \tag{92}$$

The weighted wedge product for a rank 2 temperament is (Miller 2006)

$$T_{ij} = M_{i0}M_{j1} - M_{i1}M_{j0} \tag{93}$$

It follows that $T_{ij} = -T_{ji}$ and $T_{ii} = 0$ for all i and j. So, only elements of T_{ij} with $0 < i < j$ are independent and included in the wedgie.

For a rank r temperament the wedge product is:

$$T = M_0 \wedge M_1 \wedge \ldots \wedge M_{r-1} \tag{94}$$

I don't know of a simple formula for the elements of this.

The complement of x is written as $\bar{x}$. Taking the complement of the complement gets you back where you started, although the sign may change (Browne 2001, `TheComplement` p. 16).

$$\bar{\bar{x}} = \pm x \tag{95}$$

The absolute values of the elements of the result of the complement are unchanged. However, the way they're labeled does change. Each element of the wedge product for a rank r temperament is labeled by r integers. The complement is labeled by $n - r$ integers, where n is the number of prime intervals.

6.2 Wedgie Complexity

A wedgie complexity measure is some function of the weighted wedgie. As the wedgie encodes the important properties of the temperament class, it's natural to think that the size of the wedgie will tell us the complexity. Assuming a Euclidian metric[17], the measure of the wedgie is the inner product of T with itself (Brown 2001, `TheInteriorProduct` p. 19).

$$|T| = \sqrt{T \ominus T} \tag{96}$$

Where the complement of the inner product can be defined as (Browne, `Introduction` p. 20)

$$x \bar{\ominus} y = x \wedge \bar{y} \tag{97}$$

Strictly speaking, this may not be the true inner product (you may have to take the absolute value for example) but it's close enough to be getting on with.

So, we can define the complexity of the temperament class as the measure of the weighted wedgie.

The measure has a geometric interpretation. For a vector, it's the length. For the wedge product of two vectors, it corresponds to an area. And for three vectors, it corresponds to a volume. (Browne 2001, `TheInteriorProduct` pp. 22-23)

Ben-Israel 1992 associates a volume with a matrix. It can be calculated by exterior algebra (pp. 4-6) or as a determinant (pp. 6-9).[18] The upshot is that the measure of the weighted wedgie is proportional to the scalar complexity I defined before (Equation 76 on page 18). That explains the name "scalar complexity" as the interior product is also known as the "scalar product". The exact formula for Tenney weighting is

$$k(T) = \frac{|T|}{\sqrt{n^r}} \tag{98}$$

[14] Gene prefers to think of dual spaces rather than a complement operation.

[15] For more on this, search for "geometric algebra".

[16] For suitable definitions of "temperament class".

[17] Browne 2001 uses arbitrary metrics. You could say that the weighting determines the metric for calculating the complexity of an unweighted wedgie. I prefer to think of the weighted wedgie as the thing itself, and use a Euclidian metric.

[18] The formula for the volume of a matrix as a determinant is also stated (more clearly) in Wang et. al. 2004, p. 104.

where n is the number of prime intervals and r is the rank of the temperament.

There are other ways of determining the size of the wedgie. The simplest is to take the largest absolute value. (That is, make each element positive and take the highest.) This is analogous to the "Range" complexity for rank 2 temperaments. I show it under "Max-Abs" in Tables 6 on page 26, 9 on page 29, and 12 on page 32.

Erlich 2006 uses the sum of the absolute elements of the weighted wedgie:

$$k(M) = \sum_i \sum_{j,j>i} |M_{i0}M_{j1} - M_{i1}M_{j0}| \tag{99}$$

This is in Tables 6, 9, and 12 under "Erlich".

A special case is when a temperament is defined by a single unison vector. Examples are 3-limit equal temperaments and 5-limit rank 2 temperaments. Then, the result of equation 99 is proportional to the Tenney Harmonic Distance of the unison vector. That's the reason for summing the absolute values rather than the squares, or taking the largest absolute value.

Scalar complexity implies that the metric is Euclidian, whereas the complexity of musical intervals naturally follows a city-block metric (Tenney 1984, p. 24). Unfortunately, I don't know how to calculate areas and volumes on such a metric. It'd be nice to think that summing absolute values of the wedgie does it but I don't know of any mathematical arguments for why that should work. When you calculate the measure, different elements are treated differently — with different signs and a different number of times — but this all gets ignored when you make all the distinct elements positive and add them up. Still, it gives a general idea of the complexity.

An alternative is equation 99 divided by the number of prime intervals.

$$k_M = \frac{\sum_i \sum_{j,j>i} |M_{i0}M_{j1} - M_{j1}M_{i0}|}{\sum_i v_i} \tag{100}$$

This is the same as the mean absolute value for an equal temperament. Normalizing this way gives a complexity that's around the same size as you get by other methods, whereas the sum of the absolute values is a lot bigger. Now I've worked out scalar complexity I'm not so happy with this normalized mean absolute value wedgie complexity anymore, but for the sake of completeness it's in Tables 6, 9, and 12 under "Norm-MAV".

6.3 Duality

A regular temperament class can be defined as a set of equal temperament mappings. It can also be defined as a set of vanishing intervals, called "unison vectors".[19] The wedge product of the set of unison vectors is the complement of the wedge product of the set of equal mappings (Smith 2006). Hence, anything you can calculate from one you can also calculate from the other.

Unison vectors are weighted differently to mappings. You multiply each prime by its buoyancy, hence buoyancies become weights. If the unison vector is q, the weighted unison vector, Q, is something like

$$Q_i = q_i b_i \tag{101}$$

Let T_Q be the wedge product of the weighted unison vectors. Then, a scalar complexity for Tenney weighting is similar to

$$k(T_Q) = \frac{|T_Q|}{\sqrt{n^r} \prod_i b_i} \tag{102}$$

The same principle of duality means we can also write a scalar complexity in terms of Q by analogy with Equation 76 on page 18.

$$k(Q) = \frac{\sqrt{|\langle Q^2 \rangle_V|}}{\prod_i b_i} \tag{103}$$

6.4 Wedgie Error

As the wedgie tells us everything about the temperament class, it'd be nice to be able to calculate the optimal error directly from the wedgie. Unfortunately that isn't as easy to do as for the complexity.

As scalar complexity can be calculated from the wedgie, calculating the TOP-RMS error amounts to finding the scalar badness (with $\epsilon = 0$). You can use a wedge product to calculate Equation 79 on page 18. However, the vectors you take the wedge product of are the columns of $M - V \langle M \rangle_V$, not the usual weighted wedgie.

Finding the dual of scalar badness is also difficult. From Equation 48 on page 13 we know that the Tenney weighted badness of a temperament with a single unison vector can be written

$$B\left(\frac{n}{d}\right) = \log\left(\frac{n}{d}\right) \tag{104}$$

[19] Erlich 2006 discusses unison vectors but calls them "vanishing intervals". Gene calls them "commas". The term "unison vector" comes from Fokker 1969. He applied the concept (under the name "vecteur d'homophonie") to equal temperaments in Fokker 1947. An earlier reference to similar ideas is Tanaka 1890. I don't think he gave them a name, but my German isn't that hot.

That is, the unweighted size of the interval. This works because the denominator of Equation 48 is a kind of complexity. Using a unison vector instead of a ratio,

$$B(Q) = \sum_i Q_i \tag{105}$$

As TOP-RMS involves ranges rather than sums, a good guess for the scalar badness is

$$B(Q) = \frac{\langle Q \rangle}{\prod_i b_i} \tag{106}$$

implying a TOP-RMS error of

$$\sqrt{\langle e_{\text{opt}}^2 \rangle} = \frac{\langle Q \rangle}{\sqrt{\langle Q^2 \rangle}} \tag{107}$$

Well, the good news is that this does seem to work. At least, I checked it for the temperament classes in Table 4 on page 24 with Tenney weighting. The problem comes when you generalize it for multiple unison vectors (and arbitrary weights):

$$B(Q) = \frac{\sqrt{\left|\langle Q \rangle_V^2\right|}}{\prod_i b_i} \tag{108}$$

This expands as

$$B(Q) = \frac{\sqrt{\left|\frac{Q^T V V^T Q}{V^T V V^T V}\right|}}{\prod_i b_i} \tag{109}$$

The trouble is, $Q^T V$ is a column vector. So the determinant is of the general form $\left|X X^T\right|$ where X is a column vector. And, because there's only one vector involved, this is zero whenever there's more than one unison vector!

To summarize, I don't have a general formula for optimal error or badness as a function of unison vectors.

7 Conclusion

Functions of the weighted primes are simple ways to assess the error or complexity of a regular temperament. A good, simple to calculate error is the RMS, provided you optimize the scale stretch. This leads to a standard linear least squares problem. The result is the TOP-RMS error for a temperament class, as in Equation 31 on page 9 (for numerical stability) or Equation 32 on page 9 (for simplicity). These equations generalize to other weighting schemes.

For an equal temperament, Equation 32 simplifies to Equation 25 on page 8. For a rank 2 temperament class, use Equation 38 on page 11. You can also generalize these to other weighting schemes if needs be.

The best weighted complexity formula is scalar complexity, as in Equation 76 on page 18. It can be generalized to different ranks, different weighting schemes, or different equivalence intervals. The scalar complexity of an equal temperament class is roughly the number of notes to an octave.

For octave-equivalent rank 2 temperament classes, you may find it simpler to work with the standard deviation (STD) equivalents of error and complexity. STD error is given in Equation 59 on page 14 (mapping by period and generator) and Equation 60 on page 15 (paired equal temperament mappings). STD complexity is given in Equations 69 on page 17 and 70 on page 17.

If you don't optimize the scale stretch, STD error gives better results than RMS error because it focuses on smaller intervals. However, STD error is clearly wrong for silly tunings, like where all intervals are very small. STD and RMS are both wrong therefore. The "one true error" may be a combination of the two but to decide this we need more empirical evidence.

The parameterized version of scalar badness given by either Equation 81 on page 18 or Equation 82 on page 18 lets you choose the best temperament classes with a certain trade-off between error and complexity. It only requires one free parameter. Unfortunately there's no rule to tell you how to choose this parameter for a given desired error or complexity.

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A Examples

Tables 4 and 7 show some TOP errors and complexities for the 5- and 7-limit temperaments in Tables 1 and 2 of Erlich 2006. I haven't included the TOP-max period and generator because they tend to look like the TOP-RMS period and generator and you don't learn anything from the comparison.

So that you know what the temperaments are, I show some equal temperaments that they work with. The full mappings are listed separately. Sometimes,

Table 4: Figures for some 5-limit rank 2 temperaments

ETs	Name	Scalar Complexity	TOP-RMS Error (cent/oct)	TOP-RMS Period (cent)	TOP-RMS Generator (cent)
5, 8	Father	0.443	13.194	1181.303	448.909
5, 9	Bug	0.602	11.574	1200.000	260.388
7, 10, 11, 17'	Dicot	0.521	7.095	1206.410	350.456
7, 12, 19, 31	Meantone	0.711	1.582	1201.397	504.348
12, 15, 27, 39	Augmented	0.894	2.400	399.018	93.133
7, 9, 16, 23	Mavila	0.795	6.065	1208.380	523.827
7, 15, 22, 29	Porcupine	0.960	2.678	1199.562	163.891
5, 10, 15, 20	Blackwood	1.020	4.626	238.862	80.025
8, 12, 16, 20	Dimipent	1.054	3.104	299.654	99.392
12, 22, 34, 46	Srutal	1.224	0.835	599.412	104.795
19, 22, 41, 60	Magic	1.395	1.110	1201.248	380.454
11, 12, 23, 35	Ripple	1.560	2.819	1200.283	100.862
19, 34, 53, 87	Hanson	1.550	0.274	1200.166	317.050
10, 19, 29, 48	Negripent	1.581	1.690	1202.347	126.001
7, 34, 41, 75	Tetracot	1.607	0.900	1199.561	176.095
5, 22, 27, 49	Superpyth	1.701	2.112	1197.663	488.968
12, 41, 53, 65	Helmholtz	1.791	0.057	1200.075	498.295
19, 46, 65, 84	Sensipent	1.967	0.356	1199.943	443.037
12, 49, 61, 73	Passion	2.023	1.567	1197.814	98.490
31, 34, 65, 99	Wuerschmidt	2.285	0.262	1199.694	387.701
12, 60, 72, 84	Compton	2.435	0.504	100.051	15.125
7, 46, 53, 99	Amity	2.292	0.140	1199.914	339.494
22, 31, 53, 84	Orson	2.444	0.215	1200.290	271.693
34, 50, 84, 118	Vishnu	3.708	0.047	599.977	71.140
31, 56, 87, 118	Luna	4.678	0.015	1199.980	193.198

the mapping is not the best one for that number of notes to the octave. When that's the case (according to TOP-RMS) I add an apostrophe after the octave size. This helps you to know which mapping is intended. I've listed the mappings of all the rank 2 temperaments in bra/ket form outside the tables.

In addition there are tables showing different kinds of complexities and errors. These include unweighted odd-limit figures for the sake of the comparison, although I don't talk about odd-limits. There are no figures for the 9-limit.

The list of 11-limit temperaments temperaments comes from Miller 2008. The names weren't argued over for as long as those in Erlich 2006, and even some of those are reported to have changed.

The 5-limit equal temperaments involved are: $\langle 5, 8, 12|$, $\langle 7, 11, 16|$, $\langle 8, 13, 19|$, $\langle 9, 14, 21|$, $\langle 10, 16, 23|$, $\langle 11, 17, 25|$, $\langle 12, 19, 28|$, $\langle 15, 24, 35|$, $\langle 16, 25, 37|$, $\langle 17', 27, 39|$, $\langle 19, 30, 44|$, $\langle 20, 32, 47|$, $\langle 22, 35, 51|$, $\langle 23, 36, 53|$, $\langle 27, 43, 63|$, $\langle 29, 46, 67|$, $\langle 31, 49, 72|$, $\langle 34, 54, 79|$, $\langle 35, 55, 81|$, $\langle 39, 62, 91|$, $\langle 41, 65, 95|$, $\langle 46, 73, 107|$, $\langle 48, 76, 111|$, $\langle 49, 78, 114|$, $\langle 50, 79, 116|$, $\langle 53, 84, 123|$, $\langle 56, 89, 130|$, $\langle 60, 95, 139|$, $\langle 61, 97, 142|$, $\langle 65, 103, 151|$, $\langle 72, 114, 167|$, $\langle 73, 116, 170|$, $\langle 75, 119, 174|$, $\langle 84, 133, 195|$, $\langle 87, 138, 202|$, $\langle 99, 157, 230|$, $\langle 118, 187, 274|$.

The 5-limit mappings are:

father	$\|\langle 1, 2, 2|, \langle 0, -1, 1|\rangle$
bug	$\|\langle 1, 2, 3|, \langle 0, -2, -3|\rangle$
dicot	$\|\langle 1, 1, 2|, \langle 0, 2, 1|\rangle$
meantone	$\|\langle 1, 2, 4|, \langle 0, -1, -4|\rangle$
augmented	$\|\langle 3, 5, 7|, \langle 0, -1, 0|\rangle$
mavila	$\|\langle 1, 2, 1|, \langle 0, -1, 3|\rangle$
porcupine	$\|\langle 1, 2, 3|, \langle 0, -3, -5|\rangle$
blackwood	$\|\langle 5, 8, 12|, \langle 0, 0, -1|\rangle$
dimipent	$\|\langle 4, 6, 9|, \langle 0, 1, 1|\rangle$
srutal	$\|\langle 2, 3, 5|, \langle 0, 1, -2|\rangle$

Table 5: Errors for some 5-limit rank 2 temperaments

Name	Tenney Weighted			Odd-Limit	
	TOP-RMS (cent/oct)	TOP-Max (cent/oct)	STD (cent/oct)	Minimax (cent)	RMS (cent)
Father	13.194	14.131	13.402	55.866	45.614
Bug	11.574	14.177	11.572	44.413	35.609
Dicot	7.095	7.659	7.057	35.336	28.852
Meantone	1.582	1.707	1.580	5.377	4.218
Augmented	2.400	2.941	2.406	13.686	9.678
Mavila	6.065	6.552	6.023	23.045	18.078
Porcupine	2.678	3.094	2.679	9.833	7.976
Blackwood	4.626	5.667	4.648	18.045	12.760
Dimipent	3.104	3.360	3.107	15.641	11.060
Srutal	0.835	0.890	0.836	3.259	2.613
Magic	1.110	1.281	1.109	5.923	4.569
Ripple	2.819	3.325	2.818	10.509	8.492
Hanson	0.274	0.294	0.273	1.351	1.030
Negripent	1.690	1.824	1.687	7.303	5.943
Tetracot	0.900	0.970	0.900	3.073	2.504
Superpyth	2.112	2.404	2.116	7.635	5.687
Helmholtz	0.057	0.072	0.057	0.217	0.162
Sensipent	0.356	0.414	0.356	1.489	1.157
Passion	1.567	1.686	1.569	6.735	5.488
Wuerschmidt	0.262	0.310	0.262	1.431	1.072
Compton	0.504	0.617	0.504	1.955	1.382
Amity	0.140	0.152	0.140	0.473	0.383
Orson	0.215	0.240	0.215	1.006	0.800
Vishnu	0.047	0.052	0.047	0.238	0.194
Luna	0.015	0.019	0.015	0.081	0.061

magic	$\lvert\langle 1, 0, 2\rvert, \langle 0, 5, 1\rvert\rangle$
ripple	$\lvert\langle 1, 2, 3\rvert, \langle 0, -5, -8\rvert\rangle$
hanson	$\lvert\langle 1, 0, 1\rvert, \langle 0, 6, 5\rvert\rangle$
negripent	$\lvert\langle 1, 2, 2\rvert, \langle 0, -4, 3\rvert\rangle$
tetracot	$\lvert\langle 1, 1, 1\rvert, \langle 0, 4, 9\rvert\rangle$
superpyth	$\lvert\langle 1, 2, 6\rvert, \langle 0, -1, -9\rvert\rangle$
helmholtz	$\lvert\langle 1, 2, -1\rvert, \langle 0, -1, 8\rvert\rangle$
sensipent	$\lvert\langle 1, -1, -1\rvert, \langle 0, 7, 9\rvert\rangle$
passion	$\lvert\langle 1, 2, 2\rvert, \langle 0, -5, 4\rvert\rangle$
wuerschmidt	$\lvert\langle 1, -1, 2\rvert, \langle 0, 8, 1\rvert\rangle$
compton	$\lvert\langle 12, 19, 28\rvert, \langle 0, 0, -1\rvert\rangle$
amity	$\lvert\langle 1, 3, 6\rvert, \langle 0, -5, -13\rvert\rangle$
orson	$\lvert\langle 1, 0, 3\rvert, \langle 0, 7, -3\rvert\rangle$
vishnu	$\lvert\langle 2, 4, 5\rvert, \langle 0, -7, -3\rvert\rangle$
luna	$\lvert\langle 1, 4, 2\rvert, \langle 0, -15, 2\rvert\rangle$

The 7-limit equal temperaments involved are: $\langle 5, 8, 12, 14\rvert$, $\langle 7', 11, 16, 20\rvert$, $\langle 8, 13, 19, 23\rvert$, $\langle 9, 14, 21, 25\rvert$, $\langle 10, 16, 23, 28\rvert$, $\langle 12, 19, 28, 34\rvert$, $\langle 14, 22, 32, 39\rvert$, $\langle 15, 24, 35, 42\rvert$, $\langle 15', 24, 35, 43\rvert$, $\langle 16, 25, 37, 45\rvert$, $\langle 17, 27, 40, 48\rvert$, $\langle 18, 29, 42, 51\rvert$, $\langle 19, 30, 44, 53\rvert$, $\langle 20, 32, 47, 57\rvert$, $\langle 21, 33, 49, 59\rvert$, $\langle 22, 35, 51, 62\rvert$, $\langle 23, 36, 53, 64\rvert$, $\langle 24, 38, 56, 67\rvert$, $\langle 25, 40, 58, 70\rvert$, $\langle 26, 41, 60, 73\rvert$, $\langle 27, 43, 63, 76\rvert$, $\langle 29, 46, 67, 81\rvert$, $\langle 31, 49, 72, 87\rvert$, $\langle 34', 54, 79, 95\rvert$, $\langle 34, 54, 79, 96\rvert$, $\langle 36, 57, 84, 101\rvert$, $\langle 36', 57, 83, 101\rvert$, $\langle 37, 59, 86, 104\rvert$, $\langle 38', 60, 88, 107\rvert$, $\langle 39, 62, 91, 110\rvert$, $\langle 41, 65, 95, 115\rvert$, $\langle 45, 71, 104, 126\rvert$, $\langle 46, 73, 107, 129\rvert$, $\langle 48, 76, 111, 135\rvert$, $\langle 48', 76, 112, 135\rvert$, $\langle 49, 78, 114, 138\rvert$, $\langle 50, 79, 116, 140\rvert$, $\langle 53, 84, 123, 149\rvert$, $\langle 55, 87, 128, 154\rvert$, $\langle 58, 92, 135, 163\rvert$, $\langle 60, 95, 139, 168\rvert$, $\langle 65, 103, 151, 182\rvert$, $\langle 72, 114, 167, 202\rvert$, $\langle 84, 133, 195, 236\rvert$, $\langle 89, 141, 207, 250\rvert$.

The 7-limit mappings are:

blacksmith	$\lvert\langle 5, 8, 12, 14\rvert, \langle 0, 0, -1, 0\rvert\rangle$
dimisept	$\lvert\langle 4, 6, 9, 11\rvert, \langle 0, 1, 1, 1\rvert\rangle$

Table 6: Complexities for some 5-limit rank 2 temperaments

	Tenney Weighted				TW-Wedgie			Odd-
Name	Scalar	½-Range	STD	Range	Norm-MAV	Max-Abs	Erlich	Limit
Father	0.443	1.062	0.436	2.123	0.716	1.087	2.149	2
Bug	0.602	1.292	0.602	2.584	0.851	1.292	2.554	3
Dicot	0.521	1.262	0.524	2.524	0.836	1.262	2.508	2
Meantone	0.711	1.723	0.712	3.445	1.147	1.723	3.441	4
Augmented	0.894	1.893	0.892	3.786	1.265	1.902	3.795	3
Mavila	0.795	1.923	0.800	3.846	1.275	1.902	3.825	4
Porcupine	0.960	2.153	0.960	4.307	1.439	2.153	4.318	5
Blackwood	1.020	2.153	1.015	4.307	1.442	2.174	4.327	5
Dimipent	1.054	2.524	1.053	5.047	1.687	2.524	5.062	4
Srutal	1.224	2.985	1.223	5.969	1.991	2.989	5.974	6
Magic	1.395	3.155	1.397	6.309	2.101	3.155	6.303	5
Ripple	1.560	3.445	1.560	6.891	2.291	3.445	6.872	8
Hanson	1.550	3.786	1.550	7.571	2.523	3.786	7.569	6
Negripent	1.581	3.816	1.585	7.631	2.540	3.804	7.620	7
Tetracot	1.607	3.876	1.606	7.752	2.586	3.876	7.758	9
Superpyth	1.701	3.876	1.698	7.752	2.589	3.876	7.768	9
Helmholtz	1.791	4.076	1.792	8.153	2.717	4.076	8.152	9
Sensipent	1.967	4.417	1.967	8.833	2.945	4.417	8.836	9
Passion	2.023	4.877	2.020	9.755	3.256	4.891	9.768	9
Wuerschmidt	2.285	5.047	2.285	10.095	3.366	5.047	10.097	8
Compton	2.435	5.168	2.436	10.336	3.444	5.168	10.331	12
Amity	2.292	5.599	2.292	11.198	3.733	5.599	11.199	13
Orson	2.444	5.709	2.444	11.417	3.805	5.706	11.415	10
Vishnu	3.708	8.833	3.708	17.666	5.889	8.833	17.667	14
Luna	4.678	10.325	4.678	20.651	6.884	10.326	20.651	17

dominant	$\lvert\langle 1, 2, 4, 2\rvert, \langle 0, -1, -4, 2\rvert\rangle$
august	$\lvert\langle 3, 5, 7, 9\rvert, \langle 0, -1, 0, -2\rvert\rangle$
pajara	$\lvert\langle 2, 3, 5, 6\rvert, \langle 0, 1, -2, -2\rvert\rangle$
semaphore	$\lvert\langle 1, 2, 4, 3\rvert, \langle 0, -2, -8, -1\rvert\rangle$
meantone	$\lvert\langle 1, 2, 4, 7\rvert, \langle 0, -1, -4, -10\rvert\rangle$
injera	$\lvert\langle 2, 3, 4, 5\rvert, \langle 0, 1, 4, 4\rvert\rangle$
negrisept	$\lvert\langle 1, 2, 2, 3\rvert, \langle 0, -4, 3, -2\rvert\rangle$
augene	$\lvert\langle 3, 5, 7, 8\rvert, \langle 0, -1, 0, 2\rvert\rangle$
keemun	$\lvert\langle 1, 0, 1, 2\rvert, \langle 0, 6, 5, 3\rvert\rangle$
catler	$\lvert\langle 12, 19, 28, 34\rvert, \langle 0, 0, 0, -1\rvert\rangle$
hedgehog	$\lvert\langle 2, 4, 6, 7\rvert, \langle 0, -3, -5, -5\rvert\rangle$
superpyth	$\lvert\langle 1, 2, 6, 2\rvert, \langle 0, -1, -9, 2\rvert\rangle$
sensisept	$\lvert\langle 1, -1, -1, -2\rvert, \langle 0, 7, 9, 13\rvert\rangle$
lemba	$\lvert\langle 2, 2, 5, 6\rvert, \langle 0, 3, -1, -1\rvert\rangle$
porcupine	$\lvert\langle 1, 2, 3, 2\rvert, \langle 0, -3, -5, 6\rvert\rangle$
flattone	$\lvert\langle 1, 2, 4, -1\rvert, \langle 0, -1, -4, 9\rvert\rangle$
magic	$\lvert\langle 1, 0, 2, -1\rvert, \langle 0, 5, 1, 12\rvert\rangle$

doublewide	$\lvert\langle 2, 1, 3, 4\rvert, \langle 0, 4, 3, 3\rvert\rangle$
nautilus	$\lvert\langle 1, 2, 3, 3\rvert, \langle 0, -6, -10, -3\rvert\rangle$
beatles	$\lvert\langle 1, 1, 5, 4\rvert, \langle 0, 2, -9, -4\rvert\rangle$
liese	$\lvert\langle 1, 3, 8, 8\rvert, \langle 0, -3, -12, -11\rvert\rangle$
cynder	$\lvert\langle 1, 1, 0, 3\rvert, \langle 0, 3, 12, -1\rvert\rangle$
orwell	$\lvert\langle 1, 0, 3, 1\rvert, \langle 0, 7, -3, 8\rvert\rangle$
garibaldi	$\lvert\langle 1, 2, -1, -3\rvert, \langle 0, -1, 8, 14\rvert\rangle$
myna	$\lvert\langle 1, -1, 0, 1\rvert, \langle 0, 10, 9, 7\rvert\rangle$
miracle	$\lvert\langle 1, 1, 3, 3\rvert, \langle 0, 6, -7, -2\rvert\rangle$
ennealimmal	$\lvert\langle 9, 15, 22, 26\rvert, \langle 0, -2, -3, -2\rvert\rangle$

The 11-limit equal temperaments involved are: $\langle 5, 8, 12, 14, 17\rvert$, $\langle 7', 11, 16, 20, 24\rvert$, $\langle 8, 13, 19, 23, 28\rvert$, $\langle 9, 14, 21, 25, 31\rvert$, $\langle 10, 16, 23, 28, 35\rvert$, $\langle 10', 16, 23, 28, 34\rvert$, $\langle 12, 19, 28, 34, 42\rvert$, $\langle 12', 19, 28, 34, 41\rvert$, $\langle 14, 22, 32, 39, 48\rvert$, $\langle 15, 24, 35, 42, 52\rvert$, $\langle 16, 25, 37, 45, 55\rvert$, $\langle 17, 27, 40, 48, 59\rvert$, $\langle 19, 30, 44, 53, 66\rvert$, $\langle 19', 30, 44, 53, 65\rvert$,

Table 7: Figures for some 7-limit rank 2 temperaments

ETs	Name	Scalar Complexity	TOP-RMS Error (cent/oct)	TOP-RMS Period (cent)	TOP-RMS Generator (cent)
5, 10, 15, 25	Blacksmith	0.935	5.393	239.445	87.031
8, 12, 16, 20	Dimisept	0.915	4.918	299.055	99.210
5, 7', 12, 17	Dominant	0.898	4.715	1195.412	496.521
9, 12, 15', 21	August	1.013	4.733	399.128	103.763
10, 12, 22, 34	Pajara	1.196	2.572	598.859	106.844
5, 14, 19, 24	Semaphore	1.336	2.755	1203.853	253.446
12, 19, 31, 50	Meantone	1.350	1.382	1201.242	504.026
12, 14, 26, 38'	Injera	1.350	3.138	600.683	94.483
9, 10, 19, 29	Negrisept	1.374	2.575	1203.503	125.975
12, 15, 27, 39	Augene	1.431	2.228	398.752	90.460
15, 19, 23, 34'	Keemun	1.396	2.583	1202.646	317.170
12, 24, 36, 48'	Catler	1.853	2.690	99.870	26.755
8, 14, 22, 36'	Hedgehog	1.705	2.780	599.619	164.248
5, 22, 27, 49	Superpyth	1.760	1.917	1197.067	488.512
19, 27, 46, 65	Sensisept	1.886	1.323	1199.714	443.277
10, 16, 26, 36'	Lemba	1.890	3.200	601.479	232.661
7', 15, 22, 37	Porcupine	1.726	2.531	1197.839	162.587
7', 19, 26, 45	Flattone	1.829	2.117	1203.646	507.759
19, 22, 41, 60	Magic	1.799	1.074	1201.082	380.695
18, 22, 26, 48	Doublewide	1.793	2.483	600.047	325.744
14, 15, 29	Nautilus	1.802	3.248	1202.199	82.657
10, 17, 27, 37	Beatles	1.914	2.301	1196.642	354.908
17, 19, 36, 55	Liese	1.967	2.219	1201.571	568.338
5, 26, 31, 36	Cynder	2.188	1.425	1200.937	232.375
22, 31, 53, 84	Orwell	2.257	0.748	1200.021	271.513
12, 29, 41, 53	Garibaldi	2.341	0.726	1200.125	497.967
27, 31, 58, 89	Myna	2.285	0.952	1199.344	309.976
10, 31, 41, 72	Miracle	2.444	0.515	1200.822	116.755
27, 45, 72	Ennealimmal	4.724	0.030	133.336	49.021

$\langle 22, 35, 51, 62, 76|$,
$\langle 26, 41, 60, 73, 90|$,
$\langle 29, 46, 67, 81, 100|$,
$\langle 31, 49, 72, 87, 107|$,
$\langle 34', 54, 79, 95, 117|$,
$\langle 36', 57, 83, 101, 124|$,
$\langle 39', 62, 91, 110, 136|$,
$\langle 43, 68, 100, 121, 149|$,
$\langle 46, 73, 107, 129, 159|$,
$\langle 50, 79, 116, 140, 173|$,
$\langle 53', 84, 123, 149, 184|$,
$\langle 56, 89, 130, 157, 194|$,
$\langle 58, 92, 135, 163, 201|$,
$\langle 63, 100, 146, 177, 218|$,
$\langle 68, 108, 158, 191, 235|$.

$\langle 24, 38, 56, 67, 83|$,
$\langle 27, 43, 63, 76, 94|$,
$\langle 29', 46, 68, 82, 101|$,
$\langle 33, 52, 76, 92, 114|$,
$\langle 34, 54, 79, 96, 118|$,
$\langle 37, 59, 86, 104, 128|$,
$\langle 41, 65, 95, 115, 142|$,
$\langle 45, 71, 104, 126, 155|$,
$\langle 49, 78, 114, 138, 170|$,
$\langle 53, 84, 123, 149, 183|$,
$\langle 55, 87, 128, 154, 190|$,
$\langle 57, 90, 132, 160, 197|$,
$\langle 60, 95, 139, 168, 207|$,
$\langle 65', 103, 151, 182, 224|$,

The 11-limit mappings are:

dominant	$\|\langle 1, 2, 4, 2, 1\|, \langle 0, -1, -4, 2, 6\|\rangle$
injera	$\|\langle 2, 3, 4, 5, 6\|, \langle 0, 1, 4, 4, 6\|\rangle$
augene	$\|\langle 3, 5, 7, 8, 10\|, \langle 0, -1, 0, 2, 2\|\rangle$
hedgehog	$\|\langle 2, 4, 6, 7, 8\|, \langle 0, -3, -5, -5, -4\|\rangle$
keemun	$\|\langle 1, 0, 1, 2, 0\|, \langle 0, 6, 5, 3, 13\|\rangle$
porcupine	$\|\langle 1, 2, 3, 2, 4\|, \langle 0, -3, -5, 6, -4\|\rangle$
pajara	$\|\langle 2, 3, 5, 6, 8\|, \langle 0, 1, -2, -2, -6\|\rangle$
meantone	$\|\langle 1, 2, 4, 7, 11\|, \langle 0, -1, -4, -10, -18\|\rangle$
orwell	$\|\langle 1, 0, 3, 1, 3\|, \langle 0, 7, -3, 8, 2\|\rangle$
squares	$\|\langle 1, 3, 8, 6, 7\|, \langle 0, -4, -16, -9, -10\|\rangle$
valentine	$\|\langle 1, 1, 2, 3, 3\|, \langle 0, 9, 5, -3, 7\|\rangle$
semififth	$\|\langle 1, 1, 0, 6, 2\|, \langle 0, 2, 8, -11, 5\|\rangle$

Table 8: Errors for some 7-limit rank 2 temperaments

Name	Tenney Weighted			Odd-Limit	
	TOP-RMS (cent/oct)	TOP-Max (cent/oct)	STD (cent/oct)	Minimax (cent)	RMS (cent)
Blacksmith	5.393	7.242	5.405	26.871	15.815
Dimisept	4.918	5.873	4.934	33.129	19.137
Dominant	4.715	4.771	4.733	25.345	20.163
August	4.733	5.871	4.743	24.385	16.599
Pajara	2.572	3.108	2.577	17.488	10.903
Semaphore	2.755	3.676	2.746	20.537	12.690
Meantone	1.382	1.707	1.380	5.377	3.665
Injera	3.138	3.583	3.134	17.488	11.219
Negrisept	2.575	3.193	2.568	17.848	12.189
Augene	2.228	2.941	2.235	13.686	8.101
Keemun	2.583	3.192	2.577	17.848	12.274
Catler	2.690	3.557	2.694	15.641	9.841
Hedgehog	2.780	3.108	2.782	17.488	10.602
Superpyth	1.917	2.404	1.921	9.813	6.410
Sensisept	1.323	1.612	1.324	7.489	5.053
Lemba	3.200	3.741	3.192	17.488	11.798
Porcupine	2.531	3.094	2.536	9.833	6.809
Flattone	2.117	2.542	2.111	11.702	7.652
Magic	1.074	1.281	1.073	5.923	4.139
Doublewide	2.483	3.270	2.483	17.488	10.132
Nautilus	3.248	3.486	3.242	17.848	12.629
Beatles	2.301	2.898	2.307	9.980	6.245
Liese	2.219	2.634	2.216	14.705	9.054
Cynder	1.425	1.704	1.424	5.377	3.579
Orwell	0.748	0.947	0.748	4.267	2.589
Garibaldi	0.726	0.915	0.726	4.236	2.859
Myna	0.952	1.172	0.952	5.446	3.320
Miracle	0.515	0.636	0.515	2.428	1.637
Ennealimmal	0.030	0.042	0.030	0.204	0.130

magic $|\langle 1, 0, 2, -1, 6|, \langle 0, 5, 1, 12, -8|\rangle$

meanpop $|\langle 1, 2, 4, 7, -2|, \langle 0, -1, -4, -10, 13|\rangle$

schismatic $|\langle 1, 2, -1, -3, -4|, \langle 0, -1, 8, 14, 18|\rangle$

cynder/mothra $|\langle 1, 1, 0, 3, 5|, \langle 0, 3, 12, -1, -8|\rangle$

superkleismic $|\langle 1, 4, 5, 2, 4|, \langle 0, -9, -10, 3, -2|\rangle$

myna $|\langle 1, -1, 0, 1, -3|, \langle 0, 10, 9, 7, 25|\rangle$

sensi $|\langle 1, -1, -1, -2, -8|, \langle 0, 7, 9, 13, 31|\rangle$

miracle $|\langle 1, 1, 3, 3, 2|, \langle 0, 6, -7, -2, 15|\rangle$

shrutar $|\langle 2, 3, 5, 5, 7|, \langle 0, 2, -4, 7, -1|\rangle$

tritonic $|\langle 1, 4, -3, -3, 2|, \langle 0, -5, 11, 12, 3|\rangle$

bohpier $|\langle 1, 0, 0, 0, 2|, \langle 0, 13, 19, 23, 12|\rangle$

diaschismic $|\langle 2, 3, 5, 7, 9|, \langle 0, 1, -2, -8, -12|\rangle$

rodan $|\langle 1, 1, -1, 3, 6|, \langle 0, 3, 17, -1, -13|\rangle$

wizard $|\langle 2, 1, 5, 2, 8|, \langle 0, 6, -1, 10, -3|\rangle$

B TOP-RMS Error Proofs

I've given three different formulas for TOP-RMS error: Equations 30, 31, and 32 on page 9.

Proving Equations 31 and 32 are equivalent is relatively easy as only the numerators differ, and are both determinants, so we only need to look at the bit inside the determinant. Start with that of Equation 31.

$$\begin{aligned} &\left\langle (M - V\langle M\rangle_V)^2 \right\rangle_V \\ &= \frac{(M - V\langle M\rangle_V)^T (M - V\langle M\rangle_V)}{V^T V} \\ &= \frac{M^T M}{V^T V} - \frac{M^T V\langle M\rangle_V}{V^T V} \end{aligned}$$

Table 9: Complexities for some 7-limit rank 2 temperaments

	Tenney Weighted				TW-Wedgie			Odd-
Name	Scalar	½-Range	STD	Range	Norm-MAV	Max-Abs	Erlich	Limit
Blacksmith	0.935	2.153	0.932	4.307	1.619	2.174	6.475	5
Dimisept	0.915	2.524	0.912	5.047	1.979	2.524	7.917	4
Dominant	0.898	2.435	0.894	4.870	1.989	2.455	7.956	6
August	1.013	2.137	1.011	4.274	2.076	2.148	8.305	6
Pajara	1.196	2.985	1.194	5.969	2.601	2.989	10.402	6
Semaphore	1.336	3.445	1.340	6.891	2.801	3.445	11.204	8
Meantone	1.350	3.562	1.351	7.124	2.941	3.562	11.765	10
Injera	1.350	3.445	1.352	6.891	2.979	3.445	11.918	8
Negrisept	1.374	3.816	1.379	7.631	3.031	3.804	12.125	7
Augene	1.431	4.030	1.426	8.060	3.031	4.045	12.125	9
Keemun	1.396	3.786	1.399	7.571	3.102	3.786	12.409	6
Catler	1.853	4.274	1.851	8.549	3.210	4.295	12.840	12
Hedgehog	1.705	4.307	1.704	8.614	3.297	4.307	13.190	10
Superpyth	1.760	4.589	1.756	9.177	3.608	4.602	14.431	11
Sensisept	1.886	4.631	1.885	9.261	3.615	4.631	14.459	13
Lemba	1.890	4.647	1.895	9.294	3.657	4.619	14.627	8
Porcupine	1.726	4.291	1.723	8.581	3.699	4.295	14.796	11
Flattone	1.829	4.929	1.835	9.857	3.844	4.909	15.376	13
Magic	1.799	4.274	1.800	8.549	3.884	4.274	15.536	12
Doublewide	1.793	5.047	1.793	10.095	3.899	5.047	15.596	8
Nautilus	1.802	4.307	1.806	8.614	3.906	4.307	15.623	10
Beatles	1.914	5.138	1.908	10.276	4.219	5.163	16.877	11
Liese	1.967	5.168	1.969	10.336	4.372	5.168	17.490	12
Cynder	2.188	5.524	2.190	11.049	4.612	5.523	18.448	13
Orwell	2.257	5.709	2.257	11.417	4.995	5.706	19.980	11
Garibaldi	2.341	5.618	2.341	11.236	5.073	5.619	20.292	15
Myna	2.285	6.309	2.284	12.619	5.081	6.309	20.326	10
Miracle	2.444	6.800	2.446	13.601	5.275	6.793	21.102	13
Ennealimmal	4.724	11.628	4.724	23.257	9.957	11.628	39.829	27

$$
\begin{aligned}
&-\frac{\langle M\rangle_V^T V^T M}{V^T V} + \frac{\langle M\rangle_V^T V^T V \langle M\rangle_V}{V^T V} \\
&= \left\langle M^2\right\rangle_V - \langle M\rangle_V^2 - \langle M\rangle_V^2 + \langle M\rangle_V^2 \\
&= \left\langle M^2\right\rangle_V - \langle M\rangle_V^2
\end{aligned}
$$

That's what's in the determinant of the numerator of Equation 32, so the equivalence is proved.

Proving that these formulas are equivalent to Equation 30 is more difficult. Let's start by substituting Equation 28 on page 8 into Equation 30.

$$
\begin{aligned}
\left\langle E_{\text{opt}}^2\right\rangle_V &= 1 - \left\langle M(M^T M)^{-1} M^T V\right\rangle_V \\
&= 1 - \frac{\left[M(M^T M)^{-1} M^T V\right]^T V}{V^T V} \\
&= 1 - \frac{V^T M (M^T M)^{-1} M^T V}{V^T V} \qquad (110)
\end{aligned}
$$

(As $M^T M$ is symmetric, $(M^T M)^T = M^T M$ and $[(M^T M)^{-1}]^T = (M^T M)^{-1}$.)

Now, take Equation 32 and put it into matrix form.

$$
\begin{aligned}
\left\langle E_{\text{opt}}^2\right\rangle_V &= \frac{\left|\frac{M^T M}{V^T V} - \frac{(V^T M)^T V^T M}{V^T V V^T V}\right|}{\left|\frac{M^T M}{V^T V}\right|} \\
&= \frac{\left|\frac{M^T M}{V^T V} - \frac{M^T V V^T M}{V^T V V^T V}\right|}{\left|\frac{M^T M}{V^T V}\right|}
\end{aligned}
$$

Recalling that $|AB| = |A||B|$,

$$
\begin{aligned}
\left\langle E_{\text{opt}}^2\right\rangle_V &= \frac{\left|\frac{M^T M}{V^T V}\right| \left|I - (M^T M)^{-1} V^T V \frac{M^T V V^T M}{V^T V V^T V}\right|}{\left|\frac{M^T M}{V^T V}\right|} \\
&= \left|I - \frac{(M^T M)^{-1} M^T V V^T M}{V^T V}\right| \qquad (111)
\end{aligned}
$$

Table 10: Figures for some 11-limit rank 2 temperaments

ETs	Name	Scalar Complexity	TOP-RMS Error (cent/oct)	TOP-RMS Period (cent)	TOP-RMS Generator (cent)
5, 12, 17, 29'	Dominant	1.179	4.597	1194.105	494.306
12, 14, 26	Injera	1.362	3.456	600.960	92.989
12, 15, 27, 39'	Augene	1.446	2.653	398.506	88.492
8, 14, 22, 36'	Hedgehog	1.542	2.805	600.130	164.650
15, 19', 34'	Keemun	1.485	3.579	1201.710	318.108
7', 15, 22, 37	Porcupine	1.567	2.550	1198.352	162.524
10', 12, 22, 34	Pajara	1.609	2.304	598.860	106.682
12, 19', 31, 43	Meantone	1.917	1.439	1200.772	503.355
9, 22, 31, 53	Orwell	2.050	1.151	1200.604	271.563
14, 17, 31, 45	Squares	2.205	1.449	1201.674	426.552
15, 16, 31, 46	Valentine	2.309	1.034	1200.393	77.907
7', 24, 31, 55	Semififth	2.443	1.471	1201.165	348.815
19, 22, 41, 63	Magic	2.349	1.226	1200.143	380.742
12', 19, 31, 50	Meanpop	2.416	1.239	1201.353	504.133
12, 29, 41, 53'	Schismatic	2.465	1.310	1200.199	497.711
5, 26, 31, 57	Cynder/mothra	2.523	1.371	1201.406	232.303
15, 26, 41, 56	Superkleismic	2.616	1.292	1200.176	321.894
27, 31, 58	Myna	2.610	0.851	1199.347	309.976
19', 27, 46, 65'	Sensi	2.848	1.289	1199.078	443.286
10, 31, 41	Miracle	2.786	0.484	1200.764	116.707
22, 24, 46, 68	Shrutar	2.865	1.146	599.775	52.660
29, 31, 60	Tritonic	2.907	0.999	1201.716	581.097
8, 33, 41, 49	Bohpier	3.351	1.132	1199.236	146.451
12, 34', 46, 58	Diaschismic	3.192	0.905	599.449	103.619
5, 41, 46	Rodan	3.639	0.671	1200.057	234.470
22, 50	Wizard	4.061	0.448	600.306	216.878

To show the similarity between Equations 110 and 111 I'll define two column vectors.

$$X = (M^T M)^{-1} M^T V \quad (112)$$

$$Y = \frac{M^T V}{V^T V} \quad (113)$$

With these, Equation 111 becomes

$$\left\langle E_{\text{opt}}^2 \right\rangle_V = \left| I - XY^T \right| \quad (114)$$

and Equation 110 becomes

$$\left\langle E_{\text{opt}}^2 \right\rangle_V = 1 - X^T Y \quad (115)$$

They're equivalent if the following algebraic relation holds for general column vectors X and Y:

$$\left| I - XY^T \right| = 1 - X^T Y \quad (116)$$

Felippa 2008 gives the following formula (converted to my symbols) on p. 13:

$$\left| A + \beta XY^T \right| = |A| + \beta Y^T \widetilde{A} X \quad (117)$$

where β is a scalar and $\widetilde{A}$ is the adjoint of A. In this case, it doesn't matter what an adjoint is, only that the identity matrix is its own adjoint. Obviously, $\beta = 1$. So that becomes

$$\left| I + XY^T \right| = 1 + Y^T X \quad (118)$$

That's identical to Equation 116 because $X^T Y$ and $Y^T X$ are both ways of writing the dot product of X and Y as vectors.

C STD Error Derivation

In Section 4.2, I considered the case of a rank 2 temperament class produced by two equal temperaments. Those equal temperament have weighted primes W_0 and W_1 (shown in capitals to make it clear that they're both vectors). The weighted primes for a temperament

Table 11: Errors for some 11-limit rank 2 temperaments

Name	Tenney Weighted			Odd-Limit	
	TOP-RMS (cent/oct)	TOP-Max (cent/oct)	STD (cent/oct)	Minimax (cent)	RMS (cent)
Dominant	4.597	4.964	4.620	27.685	18.933
Injera	3.456	4.056	3.451	20.137	13.345
Augene	2.653	3.367	2.663	22.386	12.773
Hedgehog	2.805	3.233	2.805	17.488	11.893
Keemun	3.579	4.414	3.574	29.159	17.216
Porcupine	2.550	3.183	2.554	19.922	11.794
Pajara	2.304	3.108	2.308	17.488	9.553
Meantone	1.439	1.746	1.438	11.022	6.584
Orwell	1.151	1.365	1.150	9.317	5.549
Squares	1.449	1.716	1.447	10.753	6.966
Valentine	1.034	1.541	1.034	8.506	4.419
Semififth	1.471	1.707	1.469	10.753	6.681
Magic	1.226	1.681	1.226	8.700	4.730
Meanpop	1.239	1.707	1.237	10.753	5.644
Schismatic	1.310	1.792	1.310	8.700	5.290
Cynder/mothra	1.371	1.704	1.370	10.753	6.457
Superkleismic	1.292	1.555	1.291	8.849	5.303
Myna	0.851	1.172	0.852	5.446	3.317
Sensi	1.289	1.612	1.290	8.435	4.976
Miracle	0.484	0.636	0.484	3.323	1.901
Shrutar	1.146	1.426	1.147	8.406	5.274
Tritonic	0.999	1.421	0.998	9.608	5.154
Bohpier	1.132	1.411	1.132	8.700	5.036
Diaschismic	0.905	1.268	0.905	5.893	3.182
Rodan	0.671	0.899	0.671	5.345	3.005
Wizard	0.448	0.641	0.448	3.052	1.585

belonging to the class are

$$w = \gamma W_0 + (1-\gamma) W_1 \tag{119}$$

where γ is a parameter specifying the tuning. When $\gamma = 1$ the tuning is the same as the first equal temperament. When $\gamma = 0$ the tuning is the same as the second equal temperament.

The weighted primes relate to the weighted mappings of equal temperaments as

$$W_{ij} = M_{ij}/M_{0j} \tag{120}$$

As the generator mapping always has zero steps to the octave, it would give an infinity if used in this context. That should please people who think that the generator mapping isn't an equal temperament mapping. But note, this only means that it isn't an *octave equivalent* equal temperament mapping. You can still call it an octave specific equal temperament where all octaves happen to approximate to no steps.

The square of the STD error for an octave-equivalent temperament is

$$\begin{aligned} e^2 &= \langle [\gamma W_0 + (1-\gamma) W_1]^2 \rangle \\ &\quad -\langle \gamma W_0 + (1-\gamma) W_1 \rangle^2 \\ &= \gamma^2 \langle W_0^2 \rangle + (1-\gamma)^2 \langle W_1^2 \rangle \\ &\quad +2\gamma(1-\gamma)\langle W_0 W_1 \rangle \\ &\quad -\gamma^2 \langle W_0 \rangle^2 - (1-\gamma)^2 \langle W_1 \rangle^2 \\ &\quad -2\gamma(1-\gamma)\langle W_0 \rangle \langle W_1 \rangle \\ &= \gamma^2 (\langle W_0^2 \rangle - \langle W_0 \rangle^2) \\ &\quad +(1-\gamma)^2 (\langle W_1^2 \rangle - \langle W_1 \rangle^2) \\ &\quad +2\gamma(1-\gamma)(\langle W_0 W_1 \rangle - \langle W_0 \rangle \langle W_1 \rangle \end{aligned}$$

$$e^2 = \gamma^2 \sigma_{W_0}^2 + (1-\gamma)^2 \sigma_{W_1}^2 + 2\gamma(1-\gamma)\sigma_{W_0 W_1} \tag{121}$$

That's a quadratic function of γ so it makes sense to write it as such

$$e^2(\gamma) = (\sigma_{W_0}^2 + \sigma_{W_1}^2 - 2\sigma_{W_0 W_1})\gamma^2$$

Table 12: Complexities for some 11-limit rank 2 temperaments

	Tenney Weighted				TW-Wedgie			Odd-
Name	Scalar	½-Range	STD	Range	Norm-MAV	Max-Abs	Erlich	Limit
Dominant	1.179	3.457	1.173	6.914	3.315	3.486	16.577	10
Injera	1.362	3.469	1.364	6.938	3.639	3.469	18.193	12
Augene	1.446	4.030	1.441	8.060	3.932	4.045	19.662	12
Hedgehog	1.542	4.307	1.543	8.614	4.038	4.307	20.189	12
Keemun	1.485	3.786	1.487	7.571	4.092	3.786	20.462	13
Porcupine	1.567	4.291	1.565	8.581	4.195	4.295	20.973	12
Pajara	1.609	4.731	1.606	9.461	4.483	4.742	22.415	16
Meantone	1.917	5.203	1.918	10.406	5.332	5.203	26.661	18
Orwell	2.050	5.709	2.051	11.417	5.704	5.706	28.520	17
Squares	2.205	6.891	2.208	13.782	5.774	6.891	28.872	16
Valentine	2.309	6.747	2.310	13.494	6.256	6.742	31.281	21
Semififth	2.443	7.364	2.446	14.727	6.461	7.364	32.306	19
Magic	2.349	6.587	2.350	13.174	6.530	6.590	32.649	20
Meanpop	2.416	7.320	2.418	14.640	6.538	7.311	32.688	23
Schismatic	2.465	5.834	2.465	11.668	6.656	5.836	33.280	20
Cynder/mothra	2.523	7.481	2.526	14.961	6.876	7.470	34.381	20
Superkleismic	2.616	6.747	2.617	13.494	7.118	6.742	35.591	21
Myna	2.610	7.227	2.609	14.453	7.312	7.227	36.562	25
Sensi	2.848	8.961	2.846	17.922	7.482	8.961	37.408	31
Miracle	2.786	7.351	2.788	14.701	7.675	7.345	38.375	22
Shrutar	2.865	8.432	2.864	16.865	7.989	8.438	39.946	22
Tritonic	2.907	7.892	2.911	15.784	8.012	7.880	40.060	22
Bohpier	3.351	8.202	3.349	16.404	8.451	8.202	42.256	26
Diaschismic	3.192	8.199	3.189	16.399	8.847	8.207	44.237	28
Rodan	3.639	11.079	3.640	22.159	9.762	11.080	48.812	30
Wizard	4.061	9.306	4.063	18.611	10.634	9.301	53.169	30

$$-2(\sigma_{W_1}^2 - \sigma_{W_0W_1})\gamma + \sigma_{W_1}^2$$

and simplify it to

$$e^2(\gamma) = \sigma_{W_0-W_1}^2\gamma^2 - 2(\sigma_{W_1}^2 - \sigma_{W_0W_1})\gamma + \sigma_{W_1}^2 \quad (122)$$

using the standard relationship

$$\begin{aligned}\sigma_{X-Y}^2 &= \langle (X-Y)^2\rangle - \langle X-Y\rangle^2 \\ &= \langle X^2\rangle + \langle Y^2\rangle - 2\langle XY\rangle \\ &\quad -\langle X\rangle^2 - \langle Y\rangle^2 + 2\langle X\rangle\langle Y\rangle \\ &= \sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY} \qquad (123)\end{aligned}$$

To optimize, set the derivative with respect to γ to zero

$$\begin{aligned}\frac{\mathrm{d}e^2}{\mathrm{d}\gamma} &= 2\sigma_{W_0-W_1}^2\gamma \\ &\quad -2(\sigma_{W_1}^2 - \sigma_{W_0W_1}) = 0 \\ \gamma &= \frac{\sigma_{W_1}^2 - \sigma_{W_0W_1}}{\sigma_{W_0-W_1}^2} \qquad (124)\end{aligned}$$

Now,[20] substitute into Equation 122 to get

$$\begin{aligned}e_{\text{opt}}^2 &= \frac{\sigma_{W_0-W_1}^2(\sigma_{W_1}^2 - \sigma_{W_0W_1})^2}{\sigma_{W_0-W_1}^4} \\ &\quad -2\frac{(\sigma_{W_1}^2 - \sigma_{W_0W_1})(\sigma_{W_1}^2 - \sigma_{W_0W_1})}{\sigma_{W_0-W_1}^2} + \sigma_{W_1}^2 \\ &= \sigma_{W_1}^2\frac{\sigma_{W_0-W_1}^2}{\sigma_{W_0-W_1}^2} - \frac{(\sigma_{W_1}^2 - \sigma_{W_0W_1})^2}{\sigma_{W_0-W_1}^2} \\ &= \frac{\sigma_{W_1}^2\sigma_{W_0}^2 + \sigma_{W_1}^4 - 2\sigma_{W_1}^2\sigma_{W_0W_1}}{\sigma_{W_0-W_1}^2} \\ &\quad -\frac{\sigma_{W_1}^4 + \sigma_{W_0W_1}^2 - 2\sigma_{W_1}^2\sigma_{W_0W_1}}{\sigma_{W_0-W_1}^2} \\ e_{\text{opt}}^2 &= \frac{\sigma_{W_0}^2\sigma_{W_1}^2 - \sigma_{W_0W_1}^2}{\sigma_{W_0-W_1}^2} \qquad (126)\end{aligned}$$

[20] The $1-\gamma$ in Equation 121 on the preceding page follows from symmetry.

$$1 - \gamma = \frac{\sigma_{W_0}^2 - \sigma_{W_0W_1}}{\sigma_{W_0-W_1}^2} \quad (125)$$

This is better written with weighted errors rather than weighted primes. Because the standard deviation of weighted errors is the same as the standard deviation of weighted primes, that's as easy as changing every W to an E. Hence Equation 60 on page 15.

D Worst Kees Error Proof

The Kees metric as considered in Section 4.3 is identical to the Tenney metric for odd harmonics. That is, if you add odd prime intervals the Kees weight of the results is the same as the Tenney weight. That means we can count on the worst weighted prime error being the worst weighted error for all harmonics, or octave equivalents. What we need, then, is a way of determining the worst weighted error for an interval between odd harmonics that don't share a factor.

The weighted error of the interval between mutually prime harmonics x and y is

$$e(x-y) = \frac{|d_x - d_y|}{b_x + b_y} \tag{127}$$

where d_x is the unweighted deviation of interval x and b_x is its buoyancy.

The absolute difference between the deviations must be the same as the difference between the highest and lowest deviations. So that gives

$$e(x-y) = \frac{\max(d_x, d_y) - \min(d_x, d_y)}{b_x + b_y} \tag{128}$$

When d_x and d_y are either both positive or both negative, then the absolute deviation of $x - y$ must be smaller than the largest absolute deviation of d_x and d_y. That means

$$e(x-y) < \frac{\max(|d_x|, |d_y|)}{\max(b_x, b_y)} \tag{129}$$

which can be re-written

$$\begin{aligned} e(x-y) &< \max(|E'_x|, |E'_y|) \\ E'_x &= \frac{d_x}{\max(b_x, b_y)} \\ E'_y &= \frac{d_y}{\max(b_x, b_y)} \end{aligned} \tag{130}$$

with E'_x as a stand-in for E_x, the weighted deviation of x.

Because E'_x has the same deviation as E_x but at least as much buoyancy, we can say

$$\begin{aligned} |E'_x| &\le |E_x| \\ |E'_y| &\le |E_y| \end{aligned} \tag{131}$$

It follows that

$$\begin{aligned} \max(|E'_x|, |E'_y|) &\le \max(|E_x|, |E_y|) \\ e(x-y) &< \max(|E_x|, |E_y|) \end{aligned} \tag{132}$$

So, when the signs of the deviations are the same, the weighted errors obey the same inequality as they do for Tenney weighting. The remaining case is of an interval between harmonics where the signs of the deviations are different. Then, the deviations combine to make the overall deviation larger than either of the originals. Using E'_x and E'_y as above, the weighted error is

$$e(x-y) = |E'_x| + |E'_y| \tag{133}$$

Using Equation 131 we know that the right hand side can't get larger. Knowing that, and that the weighted errors aren't the same, we can say

$$|E'_x| + |E'_y| < |E_x| + |E_y| \tag{134}$$

$$e(x-y) < |E_x| + |E_y| \tag{135}$$

Because that only arose from a special case of the difference between errors, and we know E_x and E_y have different signs, it can be written as

$$e(x-y) < \max(E_x, E_y) - \min(E_x, E_y) \tag{136}$$

Recalling the previous results that either adding or subtracting x and y can't increase the maximum error in any other case, we can say that

$$e(x \pm y) \le \max \begin{bmatrix} \max(E_x, E_y) - \min(E_x, E_y), \\ \max(E_x, E_y, 0), \\ -\min(E_x, E_y, 0) \end{bmatrix} \tag{137}$$

which can be simplified to

$$e(x \pm y) \le \max(E_x, E_y, 0) - \min(E_x, E_y, 0) \tag{138}$$

Now, we know that each of the harmonics was built up from prime intervals and that adding harmonics can't increase the weighted error. We also know that one of the prime errors must be zero, because we're keeping the octaves pure. That means we can place a limit on the worst Kees-weighted error for any temperament.

$$\max(e) = \max(E) - \min(E) \tag{139}$$

That's the same as Equation 52 on page 13 but without the factor of two, which is what I set out to prove.

E Parameterized Scalar Badness Proof

I defined a parameterized scalar badness in Equation 81 on page 18. Multiplying it out gives

$$\begin{aligned} B^2(M,\epsilon) &= \left|\left\langle (M-(1-\epsilon)V\left\langle M\right\rangle_V)^2\right\rangle_V\right| \\ &= \left|\left\langle \left[M-(1-\epsilon)V\frac{V^TM}{V^TV}\right]^2\right\rangle_V\right| \\ &= \left|\left\langle \left[M-(1-\epsilon)\frac{VV^T}{V^TV}M\right]^2\right\rangle_V\right| \\ &= \left|\left\langle \left[M-(1-\epsilon)AM\right]^2\right\rangle_V\right| \end{aligned}$$

where

$$A = \frac{VV^T}{V^TV} \tag{140}$$

$$\begin{aligned} B^2(M,\epsilon) &= \left|\frac{\left[M-(1-\epsilon)AM\right]^T\left[M-(1-\epsilon)AM\right]}{V^TV}\right| \\ &= \left|\frac{\left[M^T-(1-\epsilon)M^TA^T\right]\left[M-(1-\epsilon)AM\right]}{V^TV}\right| \\ &= \left|\frac{M^T\left[I-(1-\epsilon)A\right]\left[I-(1-\epsilon)A\right]M}{V^TV}\right| \end{aligned}$$

as $I = I^T$ and $A = A^T$

$$\begin{aligned} B^2(M,\epsilon) &= \left|\frac{M^T\left[II-2(1-\epsilon)A+(1-\epsilon)^2AA\right]M}{V^TV}\right| \\ &= \left|\frac{M^T\left[I-2(1-\epsilon)A+(1-\epsilon)^2A\right]M}{V^TV}\right| \end{aligned}$$

as $II = I$ and

$$XX = \frac{VV^T}{V^TV}\frac{VV^T}{V^TV} = \frac{VV^TVV^T}{V^TVV^TV} = A$$

because V^TV is a scalar and cancels.

$$\begin{aligned} B^2(M,\epsilon) &= \left|\frac{M^T\left[I-(2-2\epsilon)A+(1-2\epsilon+\epsilon^2)A\right]M}{V^TV}\right| \\ &= \left|\frac{M^T\left[I-(2-2\epsilon-1+2\epsilon-\epsilon^2)A\right]M}{V^TV}\right| \\ &= \left|\frac{M^T\left[I-(1-\epsilon^2)A\right]M}{V^TV}\right| \\ &= \left|\frac{M^TM-(1-\epsilon^2)M^TAM}{V^TV}\right| \\ &= \left|\frac{M^TM-(1-\epsilon^2)\frac{M^TVV^TM}{V^TV}}{V^TV}\right| \end{aligned}$$

from equation 140

$$\begin{aligned} &= \left|\frac{M^TM}{V^TV}-(1-\epsilon^2)\frac{M^TVV^TM}{V^TVV^TV}\right| \\ &= \left|\left\langle M^2\right\rangle_V-(1-\epsilon^2)\left\langle M\right\rangle_V^2\right| \end{aligned} \tag{141}$$

This is the formula given in Equation 82 on page 18.

As ϵ is squared in this equation, it turns out that it can be set negative. The result from either equation only depends on the magnitude of ϵ.

Like with scalar complexity, it's easy to show that this gives the same result for any mapping of the same temperament by setting $M' = MA$ where $|A| = \pm1$.

$$\begin{aligned} B^2(M',\epsilon) &= \left|\frac{M'^TM'}{V^TV}-(1-\epsilon^2)\frac{M'^TVV^TM'}{V^TVV^TV}\right| \\ &= \left|\frac{(MA)^TMA}{V^TV}-(1-\epsilon^2)\frac{(MA)^TVV^TMA}{V^TVV^TV}\right| \\ &= \left|\frac{A^TM^TMA}{V^TV}-(1-\epsilon^2)\frac{A^TM^TVV^TMA}{V^TVV^TV}\right| \\ &= \left|A^T\left[\frac{M^TM}{V^TV}-(1-\epsilon^2)\frac{M^TVV^TM}{V^TVV^TV}\right]A\right| \\ &= \left|A^T\right|\left|\frac{M^TM}{V^TV}-(1-\epsilon^2)\frac{M^TVV^TM}{V^TVV^TV}\right||A| \\ &= (\pm1)^2\left|\frac{M^TM}{V^TV}-(1-\epsilon^2)\frac{M^TVV^TM}{V^TVV^TV}\right| \\ &= \left|\frac{M^TM}{V^TV}-(1-\epsilon^2)\frac{M^TVV^TM}{V^TVV^TV}\right| \end{aligned}$$

F Symbols and Notation

$\&$ An $m\&n$ temperament is a member of the temperament class that includes the obvious mappings of the equal temperaments (in the given prime limit) with m and n notes to the octave. Sometimes the $\&$ operator can be used with the full mappings, to remove ambiguity.

A An arbitrary matrix.

$|A|$ The determinant of A (see also $|x|$).

A^T The transpose of A.

$\left\langle A\right\rangle_V$ The generalized mean of a matrix A normalized by the vector V. (See Equation 21 on page 7.)

$\left\langle A^2\right\rangle_V$ The generalized mean squared of a matrix A normalized by the vector V. (See Equation 14 on page 6.)

$\left\langle A\right\rangle_V^2$ $\left\langle A\right\rangle_V^T\left\langle A\right\rangle_V$.

α The amount of scale stretch (1 for pure octaves).

b_i The buoyancy of the ith prime interval.

$B(M)$ The badness of a regular temperament class with mapping (or mappings) M.

d The denominator of a frequency ratio.

$d(x)$ The deviation of the interval x from just intonation.

d_i The deviation of the ith prime interval from just intonation.

$e(x)$ The weighted deviation of the interval x from just intonation.

e_i The weighted deviation or error of the ith prime interval.

E The weighted errors or deviations as a column vector.

γ The tuning parameter for a rank 2 temperament.

G The generators of a regular temperament.

g_i The ith generator of a regular temperament.

h_i The size of the ith prime interval (the logarithm of the prime number, where relevant).

k_{ij} The complexity of the interval between the ith and jth primes.

$K(M)$ The complexity of a temperament class with weighted mapping M.

$\log(x)$ The logarithm (generally to base 2) of x.

n The numerator of a frequency ratio or the number of prime intervals being counted.

n_{ij} An element of the unweighted mapping for a temperament class.

m_{ij} An element of the weighted mapping for a temperament class.

M The weighted mapping as a matrix. Each row corresponds to a prime interval and each column to a generator.

M' The weighted mapping adjusted to have a generalized mean of zero.

M_{00} **or** m_{00} The number of periods to the octave.

$\max(x)$ The largest element of x.

M_g The weighted, octave equivalent generator mapping of a regular temperament.

M_i The weighted mapping of the ith generator.

$\min(x)$ The smallest element of x.

p_i The ith prime number.

$\prod_i x_i$ The product of all elements x_i.

Q A weighted unison vector.

$s(x)$ The size of the interval x.

σ_x The standard deviation of x or X (See Equation 39 on page 11).

σ_{XY} The covariance of X and Y (See Equation 40 on page 11).

$\sum_i x_i$ The sum of all elements a_i.

t_i The tempered size of the ith prime interval.

$t(x)$ The tempered size of the interval x.

T The wedgie of a temperament class.

v_i The weighted size of the ith prime interval in just intonation.

V A column vector whose elements are v_i.

w The weighted size of an arbitrary prime interval.

w_i The weighted size of the ith prime interval in a temperament.

W A column vector whose elements are w_i.

x An arbitrary number, an arbitrary interval, or something else arbitrary.

$|x|$ The absolute value of x (see also $|A|$).

$\sqrt{x}$ The square root of x.

x_i The ith element of an arbitrary vector.

X An arbitrary vector.

$\langle x \rangle$ The mean of a vector X or x. Where x is an expression, it works element by element.

X_{opt} The optimal value of X.

Y An arbitrary vector.

G Glossary

Equal mapping The mapping for an equal temperament. Mappings for higher rank temperaments can be thought of as consisting of equal mappings.

Equivalence interval An interval given a special status, usually the octave.

Generator The generators of a regular temperament or temperament class are like the prime intervals for an ideal tuning system. “The generator” for a rank 2 temperament is the one that isn’t a period, but goes with the period.

Period A generator for a regular temperament or temperament class that is either equal to the equivalence interval or equally divides the equivalence interval.

Prime interval All intervals of an ideal tuning system are generated by adding and subtracting the prime intervals (in pitch space; multiplying and dividing in frequency space).

Prime limit A set of prime intervals corresponding to prime numbers running sequentially from two to a given number.

Rank The number of generators for a regular temperament or temperament class.

Regular temperament A temperament with each interval from just intonation always approximated the same way.

RMS Root mean squared.

STD Standard deviation: the RMS relative to the mean.

Temperament A tuning system that pretends to be something else.

Temperament class Temperaments that share the same mapping from intervals in the ideal tuning.

Tenney weighting A way of weighting prime intervals according to there sizes.

TOP Tenney optimal prime: a temperament optimized according to its Tenney-weighted prime error, and that error.

Weighted mapping The mapping from primes to generators where each entry is multiplied by a weighting factor. It’s represented as a matrix where each column corresponds to an equal mapping.

Weighted primes The sizes of the prime intervals multiplied by weighting factors.

H Change Log

Removed duplicated words in the conclusion. (2008-02-07)

Rephrased ambiguous usage of "mapping" in Section 5.1. (2008-03-14)

Made Section 6.3 more vague because it’s wrong. Sorry about this. I’ll try and correct and expand on it in a future article. (2008-03-14)