Tenney-Euclidean Error Cutoffs for Equal Temperaments

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The relative Tenney-Euclidean error[1] for an equal temperament is

$$B(v,n) = \sqrt{\frac{\sum_i v_i^2 - \left(\sum_i v_i\right)^2/n}{n}} \tag{1}$$

where v_i is the weighted mapping of the ith prime, and n is the number of primes considered. You can re-arrange that to get

$$nB^2(v,n) = \sum_{i=1}^{n} v_i^2 - \left(\sum_{i=1}^{n} v_i\right)^2 /n. \tag{2}$$

Theorem 1 *The quantity $nB^2(v,n)$ always increases when n is increased by 1 and all elements of v are real.*

This is useful to know because, when you're searching for equal temperaments within a given error cutoff, you know when to stop. You can show that a subset of the mapping is already too bad and won't get any better.

To prove the theorem, replace n with $n+1$,

$$(n+1)B^2(v,n+1) = \sum_{i=1}^{n+1} v_i^2 - \left(\sum_{i=1}^{n+1} v_i\right)^2 /(n+1). \tag{3}$$

Then, take v_{n+1} out of the sums

$$(n+1)B^2(v,n+1) = v_{n+1}^2 + \sum_{i=1}^{n} v_i^2 - \left(v_{n+1} + \sum_{i=1}^{n} v_i\right)^2 /(n+1). \tag{4}$$

[1]See `http://x31eq.com/primerr.pdf` where "relative Tenney-Euclidean error" is called "scalar badness".

Expand it to get

$$(n+1)B^2(v,n+1) = v_{n+1}^2 + \sum_{i=1}^{n} v_i^2 - \frac{v_{n+1}^2 + \left(\sum_{i=1}^{n} v_i\right)^2 + 2v_{n+1}\sum_{i=1}^{n} v_i}{n+1} \tag{5}$$

and, with a bit of re-arrangement

$$(n+1)B^2(v,n+1) = \left(1 - \frac{1}{n+1}\right)v_{n+1}^2 + \sum_{i=1}^{n} v_i^2 - \frac{1}{n+1}\left(\sum_{i=1}^{n} v_i\right)^2 - \frac{2v_{n+1}}{n+1}\sum_{i=1}^{n} v_i \tag{6}$$

$$= \left(\frac{n}{n+1}\right)v_{n+1}^2 + \sum_{i=1}^{n} v_i^2 - \frac{1}{n+1}\left(\sum_{i=1}^{n} v_i\right)^2 - \frac{2v_{n+1}}{n+1}\sum_{i=1}^{n} v_i. \tag{7}$$

What we're interested in is the amount by which $nB^2(v,n)$ increases when n becomes $n+1$. That is,

$$\Delta\left(nB^2\right) = (n+1)B^2(v,n+1) - nB^2(v,n) \tag{8}$$

To find it, subtract Equation 2 from Equation 7 to get

$$\Delta\left(nB^2\right) = \left(\frac{n}{n+1}\right)v_{n+1}^2 + \sum_{i=1}^{n} v_i^2 - \frac{1}{n+1}\left(\sum_{i=1}^{n} v_i\right)^2 - \frac{2v_{n+1}}{n+1}\sum_{i=1}^{n} v_i \tag{9}$$

$$- \left[\sum_{i=1}^{n} v_i^2 - \left(\sum_{i=1}^{n} v_i\right)^2 / n\right]. \tag{10}$$

With a little re-arrangement,

$$\Delta\left(nB^2\right) = \left(\frac{n}{n+1}\right)v_{n+1}^2 - \frac{1}{n+1}\left(\sum_{i=1}^{n} v_i\right)^2 - \frac{2v_{n+1}}{n+1}\sum_{i=1}^{n} v_i + \left(\sum_{i=1}^{n} v_i\right)^2 / n \tag{11}$$

$$= \left(\frac{n}{n+1}\right)v_{n+1}^2 - \frac{2v_{n+1}}{n+1}\sum_{i=1}^{n} v_i + \left(\frac{1}{n} - \frac{1}{n+1}\right)\left(\sum_{i=1}^{n} v_i\right)^2 \tag{12}$$

$$= \left(\frac{n}{n+1}\right)v_{n+1}^2 - \frac{2v_{n+1}}{n+1}\sum_{i=1}^{n} v_i + \frac{1}{n(n+1)}\left(\sum_{i=1}^{n} v_i\right)^2. \tag{13}$$

This is a quadratic equation of form $y = ax^2 + bx + c$ where

$$\begin{aligned}
y =& \Delta\left(nB^2\right) \\
x =& v_{n+1} \\
a =& \frac{n}{n+1} \\
b =& -\frac{2}{n+1}\sum_{i=1}^{n} v_i \\
c =& \frac{1}{n(n+1)}\left(\sum_{i=1}^{n} v_i\right)^2 .
\end{aligned}$$

To find out what kind of roots it has, look at the discriminant, $b^2 - 4ac$:

$$\left(-\frac{2}{n+1}\sum_{i=1}^{n} v_i\right)^2 - 4\frac{n}{n+1}\frac{1}{n(n+1)}\left(\sum_{i=1}^{n} v_i\right)^2 = 0. \tag{14}$$

When the discriminant is zero, the equation has a single real root.[2] That means the function has either a global minimum or global maximum of zero. In this case, it's a minimum because *a* is positive. So, the change is always positive and the function can only increase as you add a prime.

[2] See, for example, Jan Gullberg, *Mathematics From the Birth of Numbers*, Norton 1996, p. 311. I call "two duplicate roots" a single root. Your mileage may vary.