Exploring Parametric Badness

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In *Primerr*[1] I defined a way of measuring the badness of a temperament class. It takes into account both the complexity (average number of notes you need) and the error (closeness to just intonation). Specifically it uses scalar complexity and TOP-RMS error. It also has a free parameter that allows you to choose the balance between complexity and error.

Here, I write more about that badness function. I talk about some useful properties that relate to rank 1 and 2 temperaments and I give examples of the best temperament classes for different prime limits and choices of parameter.

At the end there's a brief discussion about a geometric model of complexity and error, and therefore this way of measuring badness.

Contents

[1] I give code names to the articles I wrote last year. See the references.

1 Introduction

Parametric scalar badness is a way of measuring that came out of *Primerr*. I'll call it simply "badness" when I hope it's clear which kind of badness I'm talking about. It can be more precisely called "parametric badness" because I don't measure badness any other way with a free parameter.

In this article, I use the notation from *Comperr*, where the matrix M represents the unweighted mapping and W is a matrix containing the prime weights where

$$w_{ii} = \frac{1}{b_i} \tag{1}$$

$$w_{ij} = 0, i \neq j \tag{2}$$

with b_i as the buoyancy for the ith prime. H is a column vector containing the sizes of the prime intervals we're trying to approximate.

Definition 1 *The parametric scalar badness $B(E_k)$ of a temperament class defined according to the temperament-mapping matrix M approximating prime intervals H with weights W and a free parameter E_k is given by*

$$B(E_k) = \sqrt{\left| \frac{M^T W^2 M}{H^T W^2 H}(1 + E_k^2) - \frac{M^T W^2 H H^T W^2 M}{(H^T W^2 H)^2} \right|} \tag{3}$$

Where $|\ldots|$ gives the determinant.

When $E_k = 0$ this gives scalar badness as defined in *Primerr*.

$$B(0) = \sqrt{\left| \frac{M^T W^2 M}{H^T W^2 H} - \frac{M^T W^2 H H^T W^2 M}{(H^T W^2 H)^2} \right|} \tag{4}$$

Now that it's a special case of a family of ways of measuring badness, we can call it 0-badness.

2 Equal Temperaments

For an equal temperament, M is a column vector and the determinant becomes redundant. That gives us

$$B(E_k) = \sqrt{\frac{M^T W^2 M}{H^T W^2 H}(1 + E_k^2) - \frac{M^T W^2 H H^T W^2 M}{(H^T W^2 H)^2}} \tag{5}$$

It can also be written

$$B(E_k) = \sqrt{B^2(0) + E_k^2 k^2} \tag{6}$$

where $B(0)$ is the 0-badness as in Equation 4 and k is scalar complexity defined as

$$k = \sqrt{\frac{M^T W^2 M}{H^T W^2 H}} \tag{7}$$

As scalar badness is also defined as error $\times$ complexity, the parametric badness follows as

$$B(E_k) = \sqrt{E_{\text{opt}}^2 k^2 + E_k^2 k^2} \tag{8}$$

A simple theorem then follows

Theorem 1 *The parametric badness of an equal temperament with scalar complexity of k always obeys*

$$B > |E_k| k \tag{9}$$

Because the scalar complexity of an equal temperament is approximately the number of steps to the octave, we can also say

Rule of Thumb 1 *The parametric badness of an equal temperament with d steps to the octave is at least*

$$B(E_k) > \text{floor}(E_k d) \tag{10}$$

where $\text{floor}(x)$ is the largest integer no larger than x.

That may not always be true, but it's rare for k to be so different to d that it rounds to a different integer. Besides, the further k gets from d, the larger the errors must be. So there should be plenty of leeway.

Looked at another way,

Rule of Thumb 2 *If you're searching for equal temperaments within a given badness cutoff $B_{\max}$, you don't generally need to look at anything with more than $d_{\max}$ steps to the octave, where*

$$d_{\max} = \text{round}\left(\frac{B_{\max}}{E_k}\right) \tag{11}$$

and $\text{round}(x)$ is the nearest integer to x.

It follows from this that, provided $E_k > 0$, there'll always be a limit to the number of equal temperaments within a reasonable badness cutoff.[2]

[2] Because E_k is squared in Equation 3 there's no need to choose a negative value for it. So the only value that doesn't lead to a finite number of equal temperaments is $E_k = 0$.

3 Rank 2 Temperaments

Let's define a *metric* G.

$$G = \frac{W^2}{H^T W^2 H}(1 + E_k^2) - \frac{W^2 H H^T W^2}{(H^T W^2 H)^2} \tag{12}$$

Then, Equation 3 can be written as

$$B^2(M, G) = \left| M^T G M \right| \tag{13}$$

For a rank 2 temperament, M has two columns

$$M = \begin{pmatrix} M_1 & M_2 \end{pmatrix} \tag{14}$$

Substituting these into Equation 13 gives

$$B^2(M, G) = \left| \begin{pmatrix} M_1^T \\ M_2^T \end{pmatrix} G \begin{pmatrix} M_1 & M_2 \end{pmatrix} \right| \tag{15}$$

That expands out as

$$B^2(M, G) = \begin{vmatrix} M_1^T G M_1 & M_1^T G M_2 \\ M_2^T G M_1 & M_2^T G M_2 \end{vmatrix} \tag{16}$$

If G is really a metric, then it defines an *inner product* (Lay 2003, p. 428) on two column vectors X and Y that can be written as $X \cdot Y$.[3] So let's treat M_1 and M_2 as vectors.

$$B^2(M, G) = \begin{vmatrix} M_1 \cdot M_1 & M_1 \cdot M_2 \\ M_2 \cdot M_1 & M_2 \cdot M_2 \end{vmatrix} \tag{17}$$

Because G is symmetric[4],

$$M_1 \cdot M_2 = M_2 \cdot M_1 \tag{18}$$

The simple rule for the determinant of a 2×2 matrix then gives us

$$B^2(M, G) = M_1 \cdot M_1 M_2 \cdot M_2 - (M_1 \cdot M_2)^2 \tag{19}$$

[3] Strictly speaking, the inner product of a vector with itself should only be zero if the vector is itself zero (the line from a point to itself). The 0-badness as a function of vectors of arbitrary real numbers will be zero whenever the vector is a multiple of the "mapping" that gives each prime its true size in just intonation. Such vectors don't represent mappings of true temperament classes because they'll either contain irrational numbers that don't represent a mapping from just to tempered intervals, or they'll allow just intervals to map to themselves with no simplification. The latter case means the problem is incorrectly defined and the rank of the "temperament" matches the true rank of the mapping-matrix.

The vectors should always be listed with integers rather than real numbers for a true temperament mapping, In this case, the vector space becomes a *lattice*. If the problem is correctly defined, the inner product of a non-trivial vector on the lattice with itself will never be zero, even for 0-badness. I don't know if that means that even 0-badness qualifies as a lattice.

Gene Smith has written a lot about lattices and tuning theory.

The parametric badness is a true inner product, and so defines a true lattice, providing the parameter is not 0, the weights are all non-zero, and the prime intervals are linearly independent.

[4] A matrix is symmetric iff it equals its transpose. (Lay 2003, p. 449)

This is zero when $M_1 = M_2$. That makes sense because the rank 2 temperament class is identical to the equal temperament being used twice to define it, so it isn't really rank 2 at all, and the badness gives a trivial value. In fact, it's zero whenever $M_1 = \alpha M_2$ for any real number α. That is to say, it's zero whenever M_1 and M_2 are parallel.

The formula includes the badnesses for equal temperaments defined by the mappings M_1 and M_2. So let's write it accordingly, with the total badness as B, the badness of the first equal temperament as B_1, and the badness of the second equal temperament as B_2.

$$B^2 = B_1^2 B_2^2 - (M_1 \cdot M_2)^2 \tag{20}$$

Without the squares,

$$B = B_1 B_2 \sqrt{1 - \frac{(M_1 \cdot M_2)^2}{B_1^2 B_2^2}} \tag{21}$$

When M_1 and M_2 are orthogonal with respect to G – that is $M_1^T G M_2 = 0$ – the square root becomes zero. That leads to a useful rule.

Theorem 2 *If a rank 2 temperament class is composed of two equal temperaments defined by M_1 and M_2, the badness of the rank 2 class is the product of the badnesses of the two equal temperaments if M_1 and M_2 are orthogonal in badness space.*

Badness space is the *inner product space* (Lay 2003, pp. 427–442) defined by the parametric badness.

Note that the "product" relationship assumes dimensionless badness. If you're using centified badness (as I do in the examples) you take the product and divide by 1200:

$$1200 B_1 B_2 = \frac{1200 B_1 \times 1200 B_2}{1200} \tag{22}$$

Because badness is an inner product, the Cauchy-Schwarz inequality tells us that the square root term in Equation 21 is never larger than 1.(Lay 2003, p. 432) In fact, thinking in terms of an inner product space leads us straight to

$$B = B_1 B_2 \sin(\theta) \tag{23}$$

Where θ is the angle between the two mappings in the inner product space defined by G. That doesn't look like much of an advance because there's no independent definition of θ, but it does give us a rule of thumb.

Rule of Thumb 3 *If a rank 2 temperament class is composed of two equal temperaments defined by M_1 and M_2, the badness of the rank 2 class gets close to the product of the badnesses of the two equal temperaments the closer M_1 and M_2 are to being orthogonal in badness space.*

Think about the best pair of equal temperaments that defines a rank 2 temperament class. If they're the best, they must have the lowest badness. That means the product of their badnesses is the lowest of any pair of equal temperaments that define that class. For that to be the case they must also be the nearest pair to being orthogonal. Also, because they both have low badness, they must have about the same badness. Otherwise another equal temperament would get between them. So we have another rule of thumb.

Rule of Thumb 4 *The best pair of equal temperaments that defines any rank 2 temperament class are the closest to orthogonal in badness space, and the badness of each is a little more than the square root of the badness of the rank 2 class.*

Again, "square root" applies to dimensionless badness. For centified badness you multiply by 1200 and then take the square root.

4 Examples

4.1 Equal Temperaments

Table 1 shows the best 5-limit rank 1 mappings for different values of E_k. I started with 10 cents per octave (cpo) badness which gives very simple classes. These are often so out of tune you'll be hard pressed to use them as equal temperaments. I included this case to show that the badness still works with low complexities. There are two different kinds of 2-equal in the top ten. To distinguish them I called the second one 2a.[5] Interesting numbers like 7 (for the diatonic scale), 5 (the pentatonic) and 12 (the chromatic) are in the top five.

With E_k at 3 cpo 12-equal is so good that even 24-equal makes the list. As 24-equal has the same error and double the complexity, it has exactly twice the badness of 12-equal. However, it's *contorted*: not every tempered interval is an approximation of a JI interval. You can tell this because each entry in the mapping is an even number. Being even, there's a common factor of 2, and common factors signify contorsion. By some definitions[6], something with contorsion isn't a regular temperament. Hence these are lists of classes, not temperaments. What they're classes of I don't know. There like temperament classes but without having to be temperaments.

[5] Generally, an "a" signifies the second best mapping for a given number of steps to the octave. The "best" is defined as the one with the lowest STD error. That's consistent with the lowest 0-badness, and so will usually be the lowest parametric badness. Finding the one case in this article where it makes a difference if you use STD or TOP-RMS error is left as an exercise for the reader. The errors are defined in *Primerr*.

[6] For example Smith (*regular*). Note that what he calls a "temperament" I call a "temperament class".

Table 1: Centified badnesses of the best 5-limit rank 1 classes with different parameters E_k (cpo)

E_k	ID	Mapping	Badness
10	7	$\langle 7, 11, 16\|$	87.695
10	3	$\langle 3, 5, 7\|$	89.037
10	5	$\langle 5, 8, 12\|$	98.918
10	4	$\langle 4, 6, 9\|$	112.404
10	12	$\langle 12, 19, 28\|$	125.815
10	2	$\langle 2, 3, 5\|$	129.903
10	8	$\langle 8, 13, 19\|$	136.197
10	10	$\langle 10, 16, 23\|$	136.386
10	2a	$\langle 2, 3, 4\|$	138.277
10	9	$\langle 9, 14, 21\|$	141.196
3	12	$\langle 12, 19, 28\|$	51.889
3	7	$\langle 7, 11, 16\|$	57.473
3	19	$\langle 19, 30, 44\|$	67.433
3	15	$\langle 15, 24, 35\|$	83.110
3	3	$\langle 3, 5, 7\|$	84.128
3	5	$\langle 5, 8, 12\|$	86.279
3	22	$\langle 22, 35, 51\|$	88.787
3	10	$\langle 10, 16, 23\|$	97.477
3	24	$\langle 24, 38, 56\|$	103.778
3	31	$\langle 31, 49, 72\|$	105.718
1	12	$\langle 12, 19, 28\|$	39.211
1	19	$\langle 19, 30, 44\|$	40.886
1	34	$\langle 34, 54, 79\|$	48.832
1	7	$\langle 7, 11, 16\|$	54.013
1	53	$\langle 53, 84, 123\|$	54.976
1	31	$\langle 31, 49, 72\|$	59.168
1	46	$\langle 46, 73, 107\|$	61.956
1	22	$\langle 22, 35, 51\|$	63.290
1	41	$\langle 41, 65, 95\|$	65.968
1	65	$\langle 65, 103, 151\|$	69.051
0.3	53	$\langle 53, 84, 123\|$	21.613
0.3	65	$\langle 65, 103, 151\|$	30.375
0.3	34	$\langle 34, 54, 79\|$	36.478
0.3	19	$\langle 19, 30, 44\|$	36.668
0.3	118	$\langle 118, 187, 274\|$	37.060
0.3	12	$\langle 12, 19, 28\|$	37.498
0.3	106	$\langle 106, 168, 246\|$	43.227
0.3	99	$\langle 99, 157, 230\|$	43.345
0.3	46	$\langle 46, 73, 107\|$	43.692
0.3	87	$\langle 87, 138, 202\|$	47.445
0.1	53	$\langle 53, 84, 123\|$	15.572
0.1	118	$\langle 118, 187, 274\|$	16.111
0.1	171	$\langle 171, 271, 397\|$	20.483
0.1	65	$\langle 65, 103, 151\|$	24.178
0.1	152	$\langle 152, 241, 353\|$	30.558
0.1	106	$\langle 106, 168, 246\|$	31.145
0.1	236	$\langle 236, 374, 548\|$	32.222
0.1	224	$\langle 224, 355, 520\|$	32.627
0.1	99	$\langle 99, 157, 230\|$	33.077
0.1	289	$\langle 289, 458, 671\|$	33.403

Table 2: Centified badnesses of the best 7-limit rank 1 classes with different parameters E_k (cpo)

E_k	ID	Mapping	Badness
10	5	$\langle 5, 8, 12, 14\|$	99.542
10	4	$\langle 4, 6, 9, 11\|$	100.599
10	3	$\langle 3, 5, 7, 9\|$	110.584
10	7	$\langle 7, 11, 16, 19\|$	123.498
10	7a	$\langle 7, 11, 16, 20\|$	125.812
10	10	$\langle 10, 16, 23, 28\|$	128.938
10	2	$\langle 2, 3, 4, 5\|$	129.017
10	8	$\langle 8, 13, 19, 23\|$	129.274
10	2a	$\langle 2, 3, 5, 6\|$	129.534
10	3a	$\langle 3, 5, 7, 8\|$	133.023
3	12	$\langle 12, 19, 28, 34\|$	69.370
3	19	$\langle 19, 30, 44, 53\|$	77.141
3	10	$\langle 10, 16, 23, 28\|$	86.819
3	5	$\langle 5, 8, 12, 14\|$	87.106
3	22	$\langle 22, 35, 51, 62\|$	91.111
3	4	$\langle 4, 6, 9, 11\|$	93.486
3	15	$\langle 15, 24, 35, 42\|$	95.018
3	14	$\langle 14, 22, 32, 39\|$	101.733
3	9	$\langle 9, 14, 21, 25\|$	102.163
3	31	$\langle 31, 49, 72, 87\|$	102.986
1	31	$\langle 31, 49, 72, 87\|$	54.118
1	19	$\langle 19, 30, 44, 53\|$	55.509
1	12	$\langle 12, 19, 28, 34\|$	60.437
1	41	$\langle 41, 65, 95, 115\|$	60.962
1	22	$\langle 22, 35, 51, 62\|$	66.466
1	53	$\langle 53, 84, 123, 149\|$	69.970
1	27	$\langle 27, 43, 63, 76\|$	70.283
1	46	$\langle 46, 73, 107, 129\|$	76.789
1	72	$\langle 72, 114, 167, 202\|$	80.943
1	10	$\langle 10, 16, 23, 28\|$	82.088
0.3	99	$\langle 99, 157, 230, 278\|$	40.817
0.3	72	$\langle 72, 114, 167, 202\|$	42.904
0.3	31	$\langle 31, 49, 72, 87\|$	45.337
0.3	41	$\langle 41, 65, 95, 115\|$	46.785
0.3	53	$\langle 53, 84, 123, 149\|$	48.358
0.3	171	$\langle 171, 271, 397, 480\|$	52.359
0.3	19	$\langle 19, 30, 44, 53\|$	52.486
0.3	130	$\langle 130, 206, 302, 365\|$	56.142
0.3	12	$\langle 12, 19, 28, 34\|$	59.336
0.3	118	$\langle 118, 187, 274, 331\|$	60.166
0.1	171	$\langle 171, 271, 397, 480\|$	20.063
0.1	99	$\langle 99, 157, 230, 278\|$	29.688
0.1	270	$\langle 270, 428, 627, 758\|$	33.783
0.1	72	$\langle 72, 114, 167, 202\|$	37.771
0.1	342	$\langle 342, 542, 794, 960\|$	40.127
0.1	130	$\langle 130, 206, 302, 365\|$	42.422
0.1	31	$\langle 31, 49, 72, 87\|$	44.482
0.1	41	$\langle 41, 65, 95, 115\|$	45.327
0.1	140	$\langle 140, 222, 325, 393\|$	45.726
0.1	53	$\langle 53, 84, 123, 149\|$	45.975

Table 3: Centified badnesses of the best 11-limit rank 1 classes with different parameters E_k (cpo)

E_k	ID	Mapping	Badness
10	3	$\langle 3,5,7,9,11\|$	108.099
10	5	$\langle 5,8,12,14,17\|$	112.708
10	4	$\langle 4,6,9,11,13\|$	112.746
10	7	$\langle 7,11,16,19,24\|$	116.161
10	2	$\langle 2,3,5,6,7\|$	116.708
10	4a	$\langle 4,6,9,11,14\|$	117.221
10	5a	$\langle 5,8,12,14,18\|$	117.600
10	7a	$\langle 7,11,16,20,24\|$	119.291
10	8	$\langle 8,13,19,23,28\|$	123.570
10	9	$\langle 9,14,21,25,31\|$	125.840
3	12	$\langle 12,19,28,34,42\|$	80.497
3	15	$\langle 15,24,35,42,52\|$	87.563
3	22	$\langle 22,35,51,62,76\|$	91.950
3	9	$\langle 9,14,21,25,31\|$	92.461
3	14	$\langle 14,22,32,39,48\|$	93.101
3	7	$\langle 7,11,16,19,24\|$	95.666
3	8	$\langle 8,13,19,23,28\|$	96.171
3	10	$\langle 10,16,23,28,35\|$	98.693
3	7a	$\langle 7,11,16,20,24\|$	98.990
3	19	$\langle 19,30,44,53,66\|$	99.372
1	31	$\langle 31,49,72,87,107\|$	55.492
1	22	$\langle 22,35,51,62,76\|$	67.644
1	41	$\langle 41,65,95,115,142\|$	67.887
1	12	$\langle 12,19,28,34,42\|$	72.913
1	15	$\langle 15,24,35,42,52\|$	76.533
1	27	$\langle 27,43,63,76,94\|$	76.839
1	46	$\langle 46,73,107,129,159\|$	77.520
1	72	$\langle 72,114,167,202,249\|$	80.241
1	58	$\langle 58,92,135,163,201\|$	82.074
1	19	$\langle 19,30,44,53,66\|$	83.647
0.3	72	$\langle 72,114,167,202,249\|$	41.556
0.3	31	$\langle 31,49,72,87,107\|$	46.974
0.3	41	$\langle 41,65,95,115,142\|$	55.496
0.3	118	$\langle 118,187,274,331,408\|$	58.555
0.3	152	$\langle 152,241,353,427,526\|$	60.410
0.3	58	$\langle 58,92,135,163,201\|$	60.560
0.3	130	$\langle 130,206,302,365,450\|$	61.959
0.3	46	$\langle 46,73,107,129,159\|$	63.898
0.3	22	$\langle 22,35,51,62,76\|$	64.300
0.3	87	$\langle 87,138,202,244,301\|$	64.370
0.1	270	$\langle 270,428,627,758,934\|$	36.137
0.1	72	$\langle 72,114,167,202,249\|$	36.231
0.1	342	$\langle 342,542,794,960,1183\|$	39.131
0.1	152	$\langle 152,241,353,427,526\|$	42.425
0.1	31	$\langle 31,49,72,87,107\|$	46.150
0.1	224	$\langle 224,355,520,629,775\|$	48.071
0.1	118	$\langle 118,187,274,331,408\|$	48.118
0.1	130	$\langle 130,206,302,365,450\|$	49.864
0.1	41	$\langle 41,65,95,115,142\|$	54.272
0.1	311	$\langle 311,493,722,873,1076\|$	54.888

Table 4: Centified badnesses of the best 13-limit rank 1 classes with different parameters E_k (cpo)

E_k	ID	Mapping	Badness
10	4	$\langle 4,6,9,11,13,14\|$	110.681
10	5	$\langle 5,8,12,14,18,19\|$	111.958
10	7	$\langle 7,11,16,20,24,26\|$	114.609
10	3	$\langle 3,5,7,9,11,12\|$	115.717
10	5a	$\langle 5,8,12,14,17,19\|$	116.184
10	3a	$\langle 3,5,7,9,11,11\|$	117.108
10	8	$\langle 8,13,19,23,28,30\|$	118.184
10	2	$\langle 2,3,5,6,7,8\|$	119.724
10	2a	$\langle 2,3,4,5,6,7\|$	120.797
10	3b	$\langle 3,5,7,8,10,11\|$	120.833
3	9	$\langle 9,14,21,25,31,33\|$	86.218
3	12	$\langle 12,19,28,34,42,45\|$	87.542
3	8	$\langle 8,13,19,23,28,30\|$	89.182
3	10	$\langle 10,16,23,28,35,37\|$	91.339
3	15	$\langle 15,24,35,42,52,56\|$	91.736
3	7	$\langle 7,11,16,20,24,26\|$	93.241
3	19	$\langle 19,30,44,53,66,70\|$	96.292
3	19a	$\langle 19,30,44,53,65,70\|$	96.755
3	14	$\langle 14,22,32,39,48,51\|$	98.282
3	5	$\langle 5,8,12,14,18,19\|$	100.881
1	31	$\langle 31,49,72,87,107,115\|$	71.297
1	41	$\langle 41,65,95,115,142,152\|$	75.550
1	27	$\langle 27,43,63,76,94,100\|$	79.008
1	46	$\langle 46,73,107,129,159,170\|$	79.617
1	58	$\langle 58,92,135,163,201,215\|$	79.807
1	19	$\langle 19,30,44,53,66,70\|$	79.979
1	12	$\langle 12,19,28,34,42,45\|$	80.603
1	19a	$\langle 19,30,44,53,65,70\|$	80.628
1	15	$\langle 15,24,35,42,52,56\|$	81.252
1	53	$\langle 53,84,123,149,183,196\|$	82.070
0.3	72	$\langle 72,114,167,202,249,266\|$	50.715
0.3	58	$\langle 58,92,135,163,201,215\|$	57.445
0.3	87	$\langle 87,138,202,244,301,322\|$	60.326
0.3	130	$\langle 130,206,302,365,450,481\|$	61.630
0.3	41	$\langle 41,65,95,115,142,152\|$	64.637
0.3	53	$\langle 53,84,123,149,183,196\|$	64.659
0.3	31	$\langle 31,49,72,87,107,115\|$	64.881
0.3	103	$\langle 103,163,239,289,356,381\|$	66.377
0.3	46	$\langle 46,73,107,129,159,170\|$	66.433
0.3	140	$\langle 140,222,325,393,484,518\|$	70.091
0.1	270	$\langle 270,428,627,758,934,999\|$	40.146
0.1	224	$\langle 224,355,520,629,775,829\|$	46.214
0.1	72	$\langle 72,114,167,202,249,266\|$	46.454
0.1	130	$\langle 130,206,302,365,450,481\|$	49.456
0.1	58	$\langle 58,92,135,163,201,215\|$	55.047
0.1	87	$\langle 87,138,202,244,301,322\|$	55.079
0.1	311	$\langle 311,493,722,873,1076,1151\|$	55.344
0.1	494	$\langle 494,783,1147,1387,1709,1828\|$	57.015
0.1	198	$\langle 198,314,460,556,685,733\|$	57.519
0.1	140	$\langle 140,222,325,393,484,518\|$	57.836

Table 5: Centified badnesses of the best 17-limit equal temperaments with different parameters E_k (cpo)

E_k	Mapping	Badness
3	$\langle 10, 16, 23, 28, 35, 37, 41|$	85.577
3	$\langle 8, 13, 19, 23, 28, 30, 33|$	86.372
3	$\langle 12, 19, 28, 34, 42, 45, 49|$	90.216
3	$\langle 9, 14, 21, 25, 31, 33, 37|$	92.486
3	$\langle 5, 8, 12, 14, 18, 19, 21|$	95.693
3	$\langle 19, 30, 44, 53, 65, 70, 77|$	96.969
3	$\langle 14, 22, 32, 39, 48, 51, 57|$	97.783
3	$\langle 7, 11, 16, 20, 24, 26, 29|$	97.916
3	$\langle 15, 24, 35, 42, 52, 56, 62|$	98.482
3	$\langle 9a, 14, 21, 25, 31, 33, 36|$	98.685
1	$\langle 31, 49, 72, 87, 107, 115, 127|$	75.696
1	$\langle 46, 73, 107, 129, 159, 170, 188|$	75.757
1	$\langle 27, 43, 63, 76, 94, 100, 111|$	79.166
1	$\langle 22, 35, 51, 62, 76, 81, 90|$	80.089
1	$\langle 26, 41, 60, 73, 90, 96, 106|$	80.670
1	$\langle 10, 16, 23, 28, 35, 37, 41|$	80.750
1	$\langle 19, 30, 44, 53, 65, 70, 77|$	80.903
1	$\langle 29, 46, 67, 81, 100, 107, 118|$	81.775
1	$\langle 22a, 35, 51, 62, 76, 82, 90|$	82.742
1	$\langle 41, 65, 95, 115, 142, 152, 168|$	82.900
0.3	$\langle 72, 114, 167, 202, 249, 266, 294|$	48.165
0.3	$\langle 46, 73, 107, 129, 159, 170, 188|$	61.755
0.3	$\langle 58, 92, 135, 163, 201, 215, 237|$	66.342
0.3	$\langle 111, 176, 258, 312, 384, 411, 454|$	66.916
0.3	$\langle 121, 192, 281, 340, 419, 448, 495|$	68.586
0.3	$\langle 103, 163, 239, 289, 356, 381, 421|$	68.598
0.3	$\langle 87, 138, 202, 244, 301, 322, 356|$	69.294
0.3	$\langle 94, 149, 218, 264, 325, 348, 384|$	69.297
0.3	$\langle 140, 222, 325, 393, 484, 518, 572|$	69.536
0.3	$\langle 31, 49, 72, 87, 107, 115, 127|$	69.680
0.1	$\langle 72, 114, 167, 202, 249, 266, 294|$	43.656
0.1	$\langle 270, 428, 627, 758, 934, 999, 1104|$	53.665
0.1	$\langle 183, 290, 425, 514, 633, 677, 748|$	55.398
0.1	$\langle 140, 222, 325, 393, 484, 518, 572|$	57.164
0.1	$\langle 311, 493, 722, 873, 1076, 1151, 1271|$	57.662
0.1	$\langle 224, 355, 520, 629, 775, 829, 916|$	58.662
0.1	$\langle 111, 176, 258, 312, 384, 411, 454|$	59.084
0.1	$\langle 121, 192, 281, 340, 419, 448, 495|$	59.424
0.1	$\langle 46, 73, 107, 129, 159, 170, 188|$	60.369
0.1	$\langle 217, 344, 504, 609, 751, 803, 887|$	60.776

Table 6: Centified badnesses of the best 19-limit equal temperaments with different parameters E_k (cpo)

E_k	Mapping	Badness
3	$\langle 9, 14, 21, 25, 31, 33, 37, 38|$	87.196
3	$\langle 12, 19, 28, 34, 42, 45, 49, 51|$	88.394
3	$\langle 10, 16, 23, 28, 35, 37, 41, 43|$	90.937
3	$\langle 14, 22, 32, 39, 48, 51, 57, 59|$	92.725
3	$\langle 9a, 14, 21, 25, 31, 33, 36, 38|$	93.187
3	$\langle 15, 24, 35, 42, 52, 56, 62, 64|$	93.534
3	$\langle 8, 13, 19, 23, 28, 30, 33, 35|$	93.709
3	$\langle 8a, 13, 19, 23, 28, 30, 33, 34|$	93.736
3	$\langle 7, 11, 16, 20, 24, 26, 29, 30|$	94.962
3	$\langle 5, 8, 12, 14, 18, 19, 21, 22|$	95.426
1	$\langle 27, 43, 63, 76, 94, 100, 111, 115|$	75.381
1	$\langle 31, 49, 72, 87, 107, 115, 127, 132|$	77.603
1	$\langle 26, 41, 60, 73, 90, 96, 106, 110|$	78.401
1	$\langle 29, 46, 67, 81, 100, 107, 118, 123|$	78.629
1	$\langle 19, 30, 44, 53, 65, 70, 77, 80|$	80.538
1	$\langle 12, 19, 28, 34, 42, 45, 49, 51|$	81.551
1	$\langle 46, 73, 107, 129, 159, 170, 188, 195|$	81.606
1	$\langle 41, 65, 95, 115, 142, 152, 168, 174|$	81.853
1	$\langle 15, 24, 35, 42, 52, 56, 62, 64|$	83.254
1	$\langle 9, 14, 21, 25, 31, 33, 37, 38|$	83.434
0.3	$\langle 72, 114, 167, 202, 249, 266, 294, 306|$	58.895
0.3	$\langle 111, 176, 258, 312, 384, 411, 454, 472|$	66.817
0.3	$\langle 46, 73, 107, 129, 159, 170, 188, 195|$	68.812
0.3	$\langle 94, 149, 218, 264, 325, 348, 384, 399|$	68.897
0.3	$\langle 103, 163, 239, 289, 356, 381, 421, 437|$	70.267
0.3	$\langle 58, 92, 135, 163, 201, 215, 237, 247|$	70.713
0.3	$\langle 27, 43, 63, 76, 94, 100, 111, 115|$	70.811
0.3	$\langle 53, 84, 123, 149, 183, 196, 217, 225|$	70.968
0.3	$\langle 31, 49, 72, 87, 107, 115, 127, 132|$	71.744
0.3	$\langle 41, 65, 95, 115, 142, 152, 168, 174|$	71.900
0.1	$\langle 270, 428, 627, 758, 934, 999, 1104, 1147|$	51.123
0.1	$\langle 72, 114, 167, 202, 249, 266, 294, 306|$	55.267
0.1	$\langle 311, 493, 722, 873, 1076, 1151, 1271, 1321|$	55.867
0.1	$\langle 217, 344, 504, 609, 751, 803, 887, 922|$	58.523
0.1	$\langle 111, 176, 258, 312, 384, 411, 454, 472|$	58.971
0.1	$\langle 422, 669, 980, 1185, 1460, 1562, 1725, 1793|$	62.923
0.1	$\langle 183, 290, 425, 514, 633, 677, 748, 777|$	63.476
0.1	$\langle 94, 149, 218, 264, 325, 348, 384, 399|$	63.563
0.1	$\langle 354, 561, 822, 994, 1225, 1310, 1447, 1504|$	63.858
0.1	$\langle 103, 163, 239, 289, 356, 381, 421, 437|$	63.952

3 cpo is still fairly large (around 9 cents error for ratios with numbers up to 8) so some over-simple mappings like 3-equal still make the list. But, generally speaking, these are either good equal temperaments or good scales. You can even argue that the 3 means three notes to a triad.

1 cpo gives a good trade-off between complexity and error. Probably your favorite 5-limit equal temperament is in this list. 53-equal is so good here that it also makes the list as a contorted 106-equal. Both are still there at 0.1 cpo, as well as 236, the contorted 118.

Table 2 shows the same lists for the 7-limit. The 3 cpo list doesn't show a 7. That implies that diatonic scales don't work so well with 7-limit harmony although pentatonics and 12 note chromatics do. Once again, your favorite equal temperament is probably in the 1 cpo list.

171-equal has significantly lower 0.1-cpo badness than anything else. It also makes the list in contorted form with 270 notes to the octave.

The 11-limit lists are in Table 3 and the 13-limit lists in 4. 19-equal doesn't map 11:8 very well so it barely makes the 11-limit lists and then tends to come up twice in the 13-limit lists, with different mappings of 11. There's less difference in badness between the best and worst classes in any list now, so there are no more contorted mappings.

Tables 5 and 6 show the 17- and 19-limit lists. Because the mappings are getting wide I left out the "ID" column. The lesser mapping with a given number of steps to the octave is still denoted with an "a" but it's inside the mapping. The interval 19:1 comes out as about 33.983 steps of 8-equal. However, because of scale shrinkage, it gets mapped 35 steps in the optimal temperament — more than a step away. This is a rare case where rounding the mapping of each prime up and down doesn't include the optimal mapping. (Of course, the badnesses shown are almost the same so it doesn't really matter.)

The more primes you look at the less difference there is in badness between different classes in a given list. The reason is that there are a lot more primes to get right. The chances are that a good match for one will be balanced by a less good match for another.

4.2 Rank 2 Temperaments

Table 7 shows the best 5-limit linear temperaments, along with classes for things like linear temperaments that have contorsion. I've tried to use existing names where I can find them.[7] The "ETs" column shows the best pair of equal temperaments consistent with this rank 2 class. You can lookup the mappings in table 1.

For each value of E_k, the best rank 2 class is composed of the two best equal temperaments for the same E_k. Sometimes the same rank 2 temperament comes up in different lists, but composed of different equal temperaments. That's nothing to worry about! All it means is that the order of equal temperaments also depends on E_k.

Table 7: Centified badnesses of the best 5-limit rank 2 classes with different parameters E_k (cpo)

E_k	Name	ETs	Badness	θ
3	Meantone	7 & 12	2.411	76°
3	Augmented	12 & 15	3.437	73°
3	Srutal	12 & 22	3.813	83°
3	Porcupine	7 & 15	3.861	76°
3	Dicot	3 & 7	4.013	85°
1	Meantone	12 & 19	1.330	85°
1	Srutal	12 & 34	1.595	89°
1	Hanson	19 & 34	1.607	75°
1	Helmholtz	12 & 53	1.794	87°
1	Magic	19 & 22	2.084	75°
0.3	Helmholtz	53 & 65	0.547	89°
0.3	Hanson	34 & 53	0.629	73°
0.3	Amity	53 & 99	0.759	77°
0.3	Orson	53 & 84	0.902	78°
0.3	Würschmidt	34 & 65	0.910	80°
0.1	Helmholtz	53 & 118	0.206	81°
0.1	Amity	53 & 152	0.395	85°
0.1	Vishnu	118 & 152	0.410	87°
0.1	Co-Helmholtz	65 & 171	0.412	88°
0.1	Vulture	53 & 270	0.440	90°

I've also included the badness-space angle between the two defining equal temperaments. It isn't that interesting, but neither does it take up much space.

For 1 cpo badness, the best three equal temperaments all give meantone. And for 0.1 cpo badness, all of the top ten equal temperaments except 99- and 152-equal are consistent with helmholtz (either directly or with contorsion). The best pair are almost orthogonal. This shows that good rank 2 classes tend to be associated with good equal temperaments where "good" means having low badness with the same value for E_k.

Orson, in the 0.3 cpo list, involves a mapping for 84-equal that doesn't make the top ten. That mapping is:

$\langle 84, 133, 195|$

Helmholtz is so good as a 0.1 cpo temperament that it comes up again in contorted form. I call this "co-helmholtz". There are, in fact, three distinct temperament-like classes corresponding to a helmholtz (or schismatic[8]). with contorsion of 2: 65 & 171, 16 & 118, and 53 & 236. (106 & 118 is the 5-limit subset of *hemischismic*.) They all have the same complexity (double that of helmholtz) and error (the same as helmholtz)

[7] Many originate with Smith (*sevnames*). A few obscure ones were identified by Miller (2009).

[8] In these table, "helmholtz", "garibaldi", "pontiac", and "cassandra" are all kinds of schismatic

Table 8: Centified badnesses of the best 7-limit rank 2 classes with different parameters E_k (cpo)

E_k	Name	ETs	Badness	θ
3	Meantone	12 & 19	4.459	89°
3	Pajara	10 & 12	4.727	70°
3	Dominant	5 & 12	5.018	85°
3	Dimisept	4 & 12	5.270	77°
3	Augene	12 & 15	5.346	77°
1	Meantone	19 & 31	2.303	67°
1	Magic	19 & 41	2.640	69°
1	Miracle	31 & 41	2.749	89°
1	Orwell	22 & 31	2.818	70°
1	Garibaldi	12 & 41	2.892	70°
0.3	Ennealimmal	72 & 99	1.424	77°
0.3	Miracle	31 & 72	1.456	64°
0.3	Hemiwürschmidt	31 & 99	1.524	81°
0.3	Hemififths	41 & 99	1.589	87°
0.3	Catakleismic	53 & 72	1.624	70°
0.1	Ennealimmal	99 & 171	0.493	83°
0.1	Sesquiquartififths	130 & 171	0.707	85°
0.1	Tertiaseptal	31 & 171	0.734	81°
0.1	Pontiac	53 & 171	0.769	90°
0.1	Enneadecal	152 & 171	0.781	86°

Table 9: Centified badnesses of the best 11-limit rank 2 classes with different parameters E_k (cpo)

E_k	Name	ETs	Badness	θ
3	Augene	12 & 15	5.790	80°
3	August'	9 & 12	5.947	74°
3	Meantone'	7 & 12	5.952	68°
3	Pajara	12 & 22	6.084	81°
3	Porcupine	15 & 22	6.171	67°
1	Miracle	31 & 41	3.096	80°
1	Orwell	22 & 31	3.126	88°
1	Valentine	15 & 31	3.322	70°
1	Meantone	12 & 31	3.360	85°
1	Myna	27 & 31	3.428	75°
0.3	Miracle	31 & 72	1.587	77°
0.3	Unidec'	72 & 118	1.978	77°
0.3	Harry	58 & 72	1.995	72°
0.3	Octoid	72 & 152	2.054	79°
0.3	Wizard	22 & 72	2.191	80°
0.1	Hemiennealimmal	72 & 270	0.993	65°
0.1	Vishnu"	152 & 270	1.234	75°
0.1	Octoid	72 & 152	1.237	75°
0.1	Miracle	31 & 72	1.378	81°
0.1	—	152 & 342	1.383	90°

and so they have the same badness. They also have the same *wedgie*[9] and so the same octave-equivalent mapping. As they have different melodic structures there is an argument for listing them all, in which case they'd tie for 4th place, and vulture would drop out of the list. If you're interested in contorted temperament-like things you'll want to check all of them, but once you recognize the contorsion you can tell there are other classes out there with the same badness. You may not be interested in contorted things, in which case it's better to leave everything with contorsion out of the list. You can be reassured that this is the only example of rank 2 contorsion in this article.

Vulture depends on 270-equal, which doesn't have a mapping in Table 1 because all those helmholtz subsets pushed it out. However it's the 5-limit subset of the 7-limit mapping given in Table 2 and other places.

Now to the 7-limit rank 2 classes in Table 8. Each list makes some sense: 3 cpo for rough and ready temperaments, 1 cpo for the more accurate ones, 0.3 cent for the really accurate ones, and 0.1 cpo for people who thought the 0.3 cpo list wasn't accurate enough. Once again the best rank 2 class is always composed of the best two equal temperaments for the same value of E_k.

171-equal dominates the 0.1 cpo equal temperaments so much that it gets involved with the best five 0.1 cpo rank 2 classes. Of these, Ennealimmal has by far the lowest badness (it's also top of the list for 0.3 cpo badness). The best five 0.1 cpo equal temperaments are all consistent with ennealimmal (342-equal is 171-equal with contorsion).

Enneadecal involves an equal temperament that didn't make the top 10. And once again you can get its mapping from the lists for higher limits.

The 11-limit results are in Table 9. Sometimes I don't have a name for a temperament class, but I do have a name for a subset from a lower prime limit. So I show that name along with an apostrophe for each prime I excluded. This indicates that the meantone in the 3 cpo list has a different mapping for 11:8 to the more usual meantone in the 1 cpo list.

In the 1 cpo table, miracle pushes out orwell although the best two 1 cpo equal temperaments from Table 3 are 31- and 22-equal. Miracle has a smaller badness space angle. *C'est la vie.* But there's still a good correlation between good equal and rank 2 temperaments: each rank 2 class in each top five is composed of two equal temperaments from the corresponding top ten. This is also true in the 13-limit (Table 10) and the 17-limit (Table 11). Like with equal temperaments, the range of badnesses is getting smaller as the prime limit gets larger.

One notable absence from the 13- and 17-limit lists is *mystery* (58 & 87). That's a little surprising because 58-

[9] A way of uniquely identifying a temperament class. See *Primerr*, Smith (*wedgie*) and other places.

Table 10: Centified badnesses of the best 13-limit rank 2 classes with different parameters E_k (cpo)

E_k	Name	ETs	Badness	θ
3	Negrisept”	9 & 10	5.977	66°
3	—	8 & 9	6.106	72°
3	August”	9 & 12	6.196	80°
3	Dominant”	7 & 12	6.296	68°
3	Porcupine”’	8 & 15	6.368	69°
1	Myna	27 & 31	4.216	64°
1	Miracle	31 & 41	4.280	72°
1	Meantone	12 & 31	4.296	64°
1	Orwell	31 & 53	4.501	67°
1	Cassandra	12 & 41	4.588	65°
0.3	Harry	58 & 72	2.269	69°
0.3	Tritikleismic”	72 & 87	2.548	88°
0.3	Miracle’	31 & 72	2.557	69°
0.3	Catakleismic”	53 & 72	2.718	84°
0.3	Miracle’	41 & 72	2.728	87°
0.1	—	224 & 270	1.532	82°
0.1	Hemiennealimmal’	72 & 270	1.542	83°
0.1	—	130 & 270	1.628	80°
0.1	Octoid’	72 & 224	1.754	79°
0.1	—	87 & 270	1.782	75°

Table 11: Centified badnesses of the best 17-limit rank 2 classes with different parameters E_k (cpo)

E_k	Name	ETs	Badness	θ
3	Octokeidecal”’	8 & 10	6.156	88°
3	—	8 & 9	6.176	68°
3	Negrisept”’	9 & 10	6.292	73°
3	Dimisept”’	8 & 12	6.303	76°
3	Decimal”’	10 & 14	6.359	66°
1	Sensi”	27 & 46	4.450	63°
1	Miracle’	10 & 31	4.574	64°
1	Unidec”’	26 & 46	4.606	65°
1	Shrutar”	22 & 46	4.738	70°
1	Negrisept”’	10 & 19	4.751	61°
0.3	Unidec”’	46 & 72	2.477	88°
0.3	Miracle”	72 & 103	2.601	71°
0.3	Harry	58 & 72	2.621	80°
0.3	—	72 & 111	2.650	81°
0.3	—	72 & 121	2.740	85°
0.1	—	72 & 183	1.918	72°
0.1	Hemiennealimmal”	72 & 270	1.952	89°
0.1	—	72 & 140	2.062	82°
0.1	—	72 & 311	2.097	89°
0.1	Octoid”	72 & 224	2.132	88°

and 87-equal are two of the best three equal temperaments for the 13-limit with 0.3 cpo badness. Sometimes it comes top of odd-limit lists but as many intervals are equally complex it scores lower for scalar complexity. The odd limit interpretation makes a lot of sense: you get each 15-limit chord 29 times over in a 58 note scale. In practice that does make it simpler and more efficient than other temperaments of similar error. So you may think it should be in the top five. The moral is that you shouldn’t take the rankings too seriously. There isn’t enough space in this article but, if you’re looking for temperament classes, you should check a longer list in case it has something that interests you. (Mystery happens to be number 7 in the 13-limit 0.3 cpo list. It’s the best class that doesn’t include 72-equal and has a badness-space angle of 84°.)

When we get to the 19-limit, in Table 12, some strange things start happening. The first few 3 cpo classes are composed of different mappings of the same equal temperament. One of those mappings is missing from Table 6:

$\langle 10, 16, 23, 28, 35, 37, 41, 42|$

There’s another missing equal temperament for the 0.3 cpo list:

$\langle 68, 108, 158, 191, 235, 252, 278, 289|$

This is, in fact, the 17th best 19-limit equal temperament according to 0.3 cpo badness. That makes this list the hardest one to find, even though this pair of equal temperaments doesn’t have the lowest badness space angle in the tables. The reason is that so many 19-limit temperaments — of both ranks 1 and 2 — have roughly the same badness that the good ones don’t stand out very much.

The best 19-limit rank 2 classes also tend not to be supersets of the best lower-limit classes for smaller errors. The 19-limit is really a different world. The best equal temperaments are, however, largely familiar.

5 Alternative Formulations

In *Primerr*, I defined parametric badness differently.

Definition 2 *The parametric scalar badness $B(\epsilon)$ with a free parameter ϵ and other symbols as in Definition 1 is given by*

$$B(\epsilon) = \sqrt{\left|\frac{M^TW^2M}{H^TW^2H} - (1-\epsilon^2)\frac{M^TW^2HH^TW^2M}{(H^TW^2H)^2}\right|} \tag{24}$$

Equation 4 still holds when $\epsilon = 0$. When $\epsilon = 1$ the result is scalar complexity

$$B(\epsilon = 1) = k = \sqrt{\left|\frac{M^TW^2M}{H^TW^2H}\right|} \tag{25}$$

Table 12: Centified badnesses of the best 19-limit rank 2 classes with different parameters E_k (cpo)

E_k	Name	ETs	Badness	θ
3	—	9 & 9a	6.138	65°
3	—	10 & 10a	6.287	60°
3	—	8 & 8a	6.326	60°
3	Negrisept””	9 & 10	6.349	74°
3	Bug”””	5 & 9	6.385	67°
1	Sensi’”	19 & 27	4.508	63°
1	Superkleismic’”	26 & 41	4.746	63°
1	Injera’”	12 & 26	4.761	63°
1	Myna”	27 & 31	4.791	79°
1	Negrisept””	19 & 29	4.811	66°
0.3	Unidec””	46 & 72	3.216	72°
0.3	—	72 & 111	3.264	84°
0.3	Wizard’”	72 & 94	3.365	84°
0.3	—	68 & 72	3.370	68°
0.3	Miracle’”	41 & 72	3.402	75°
0.1	Hemiennealimmal’”	72 & 270	2.354	89°
0.1	—	270 & 311	2.366	84°
0.1	—	111 & 270	2.421	75°
0.1	Vulture””	217 & 270	2.440	78°
0.1	—	111 & 311	2.516	66°

Theorem 3 *The badness in Equation 24 can also be written as*

$$B(\epsilon) = \sqrt{\left|\frac{M'^T M'}{H^T W^2 H}\right|} \tag{26}$$

where

$$M' = WM - (1-\epsilon) WH \frac{H^T W^2 M}{H^T W^2 H} \tag{27}$$

This is the equivalence I proved in Appendix E of *Primerr*. It also defines badness space vectors. (The $H^T W^2 H$ term is simply a constant.)

Theorem 4 *The badnesses in Equation 3 and Equation 24 can be related as*

$$B(E_k) = \frac{B(\epsilon)}{(1-\epsilon^2)^{\frac{r}{2}}} \tag{28}$$

where r is the rank of the temperament and

$$\epsilon = \frac{E_k}{\sqrt{1+E_k^2}} \tag{29}$$

Knowing this, you can use the $B(\epsilon)$ formulas to calculate $B(E_k)$. If you want to know E_k given ϵ it follows as

$$E_k = \frac{\epsilon}{\sqrt{1-\epsilon^2}} \tag{30}$$

Because "badness" doesn't measure anything in real life, getting the figures precise doesn't matter much. Given that, and the fact that ϵ and E_k are typically small, we can forget about the difference altogether.

Rule of Thumb 5 *For most practical cases, $B(E_k) = B(\epsilon)$ and $E_k = \epsilon$.*

Because $B(E_k) \propto B(\epsilon)$, the difference between them doesn't matter at all when you're comparing different temperament classes to see how "bad" they are relative to each other. The difference between E_k and ϵ sometimes matters a little bit.

So that it's clear whether I'm talking about E_k-badness or ϵ-badness, I'll always measure E_k in cpo and give ϵ as a dimensionless number. Remember to divide the value in cpo by 1200 for the equations to work.

To prove Theorem 4, substitute Equation 24 into Equation 28

$$\begin{aligned} B(E_k) &= \frac{\sqrt{\left|\frac{M^T W^2 M}{H^T W^2 H} - (1-\epsilon^2)\frac{M^T W^2 H H^T W^2 M}{(H^T W^2 H)^2}\right|}}{(1-\epsilon^2)^{\frac{r}{2}}} \\ &= \sqrt{\frac{\left|\frac{M^T W^2 M}{H^T W^2 H} - (1-\epsilon^2)\frac{M^T W^2 H H^T W^2 M}{(H^T W^2 H)^2}\right|}{(1-\epsilon^2)^r}} \\ &= \sqrt{\left|\frac{M^T W^2 M}{H^T W^2 H}\frac{1}{1-\epsilon^2} - \frac{M^T W^2 H H^T W^2 M}{(H^T W^2 H)^2}\right|} \end{aligned} \tag{31}$$

That matches Equation 3 where the free parameters are related as

$$1 + E_k^2 = \frac{1}{1-\epsilon^2} \tag{32}$$

from which you can derive Equation 29 and Equation 30.

6 Geometric Interpretation

I've talked above about a badness space defined by parametric badness as an inner product. We can also think about badness as a function in *complexity space* defined by the inner product

$$X \cdot Y = X^T W^2 Y \tag{33}$$

This is a normal Euclidean space with each axis scaled proportionally to its weight (so highly weighted intervals end up shorter). Gene Smith has described a similar space, but using a taxi-cab metric instead of a Euclidean one. (Smith *sevlat*)

In complexity space, the scalar complexity of an equal temperament is its distance from the origin divided by the distance of the *JI point* (the position of H) from the origin.

Lay (2003, p. 386) defines the *orthogonal component* of $\mathbf{y}$ onto $\mathbf{u}$ as

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \tag{34}$$

Then $\mathbf{y}$ can be written as

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \tag{35}$$

where $\mathbf{z}$ is a vector orthogonal to $\mathbf{u}$. You can rearrange that to give a formula for finding the component of $\mathbf{y}$ orthogonal to $\mathbf{u}$.

$$\mathbf{z} = \mathbf{y} - \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} \tag{36}$$

By this definition, the component of M orthogonal to H is

$$M - \frac{M \cdot H}{H \cdot H}H \tag{37}$$

Expanding out the scalar products, and re-ordering some products, that becomes

$$M - H\frac{H^T W^2 M}{H^T W^2 H} \tag{38}$$

Compare that with Equation 27 and you will see the following.

Theorem 5 *The 0-badness of an equal temperament defined by M is the distance from the origin to the component of M orthogonal to the position vector of the JI point H in complexity space.*

The extra factors of W in Equation 27 come from the definition of the scalar product you use to find the distance in complexity space.

If you know your Grassman Algebra, this talk of orthogonal components may remind you of wedge products.[10] It means I can give a formula that was missing from *Primerr*.

Theorem 6 *The TOP-RMS error of a temperament class defined by the weighted wedgie T and $V = WH$ is*

$$E = \frac{|T \wedge V|}{|T||V|} \tag{39}$$

where $|X|$ is the magnitude of X.

This is because the wedge product of a set of vectors always gives you something orthogonal to all of them. The 0-badness is proportional to

$$E = |T \wedge V| \tag{40}$$

Then you divide that by the scalar complexity (the magnitude of T) and normalize it to be dimensionless.

The geometrical properties of exterior algebra lead us to

Theorem 7 *The TOP-RMS error of a temperament class defined by M is the sine of the angle between the* span *of the columns of M and the JI point H in complexity space.*

[10]You can also compare with the formula given by Smith (2006). It's the same basic form but uses a different way of measuring the magnitude.

I can't find a statement of this in terms of wedge products so you'll have to be content with an analogy with the 3 dimensional vector product (Clapham 1996, p. 294) and remember that wedge products are a generalization of vector products. The "span" here is the line/plane/hyperplane containing all equal temperament mappings belonging to the temperament class and the origin. (Lay 2003, p. 35) That corresponds to all the different tunings (including silly ones) of the temperament class.

I still don't know how to define parametric badness in terms of wedge products. The geometric argument is as follows: the operation that finds the component orthogonal to H is also a projection. It means projecting onto a plane/hyperplane that passes through the origin and is orthogonal to H. All points are moved parallel to H until they hit this hyperplane. It means 0-badness removes the dimension of the space parallel to H. The parametric badness merely shrinks that dimension. Transforming to badness space means you move points closer to this hyperplane but not all the way. The higher their initial complexity, the further they end up. The more you shrink the H-dimension, the more you favor high complexity temperaments.

7 Conclusion

This parametric badness is a good way of measuring the badness of temperament classes. It means you have to choose the parameter, and it can give very different results for different choices of that parameter. That may be considered a weakness but I don't think it's reasonable to expect a single number to tell you any meaningful badness unless you specify what kind of error or complexity you're interested in. The single parameter (which relates to the optimal error) is the simplest way of doing that.

Setting the parameter to 1 cpo is a good place to start, and you can adjust it depending on the results you get. Your specific desires are likely not to be reflected in the specific numbers that come out so it's always worth generating enough temperament classes that you can look through them to find what you want. As this badness is simple to calculate and search for it can also be used as a first step to generating a shortlist based on some other way of measuring badness.

8 References

So I don't have to keep saying my name, I give code names to the articles I reference.

Gene Smith's old website is now mirrored by Carl Lumma. I don't know the original dates, and they're likely the same year, so I give them code names as well, but I include his name in the citations.

1. *Primerr*: "Prime Based Error and Complexity Measures", completed February 2008. Available at `http://x31eq.com/primerr.pdf`.

2. *Comperr*: "RMS-Based Error and Complexity Involving Composite Intervals", completed April 2008. Available at `http://x31eq.com/composite.pdf`.

3. D. C. Lay 2003. *Linear Algebra and Its Applications*, Third Edition, Pearson. (Republished by Publishing House of Electronics Industry, Beijing, 2004.)

4. C. Clapham, 1996. *Oxford Concise Dictionary of Mathematics*. OUP, second edition. (Republished by Shanghai Foreign Language Press, 2001.)

5. H. Miller, 2009. Tuning-math post January 3, 2009.

6. G. W. Smith 2006. "Re: Complete rank 2 temperament surveys" Tuning-math post January 21, 2006. Archived at `http://tech.groups.yahoo.com/group/tuning-math/message/14189`

7. G. W. Smith *regular*. `http://lumma.org/tuning/gws/regular.html`. Accessed December 30 2008.

8. G. W. Smith *sevnames*. `http://lumma.org/tuning/gws/sevnames.htm`. Accessed December 27 2008.

9. G. W. Smith *sevlat*. `http://lumma.org/tuning/gws/sevlat.htm`. Accessed January 2 2009.

10. G. W. Smith *wedgie*. `http://lumma.org/tuning/gws/wedgie.html`. Accessed December 30 2008.

9 Glossary

Badness A number that gives you some idea how bad a temperament or temperament class is. Gets higher as the error or complexity gets higher.

Complexity A way of measuring how many pitches you're likely to need to approximate harmony of a given complexity in a given temperament.

cpo Cents per octave. Convenient units to measure errors in. To get it, multiply a normalized error by 1200.

Centified The centified badness is the normalized badness multiplied by 1200. The exact units are clumsy, but it's useful when comparing with errors in cpo and generally makes the numbers more friendly.

Determinant A number assigned to a matrix. See, for example, Clapham (1996, pp. 68–69) for details.

Contorsion Something like a temperament that has intervals that don't approximate anything in JI is called *contorted* and has *contorsion*.

Error A way of measuring how close a temperament gets to JI. Generalized to temperament classes by considering optimal tunings.

Euclidean The kind of geometry you studied if nobody told you otherwise. The Euclidean distance is that measured by a ruler, or as the crow flies.

Inner product A real valued function of two vectors with certain properties. Defines an inner product space.

Just intonation An ideal system of musical tuning.

JI Just intonation.

Metric A way of measuring distances. Here, generally a matrix defining an inner product.

Orthogonal At right angles to, or the one not depending on the other.

Prime limit The prime numbers (or generalizations thereof) used in ratios for intervals of just intonation. For example, the prime limit 7 — or 7-limit — includes the primes 2, 3, 5, and 7.

Rank The rank of a temperament is the number of independent intervals used to define it. An equal temperament has a rank of 1. A rank r temperament can be defined by r equal temperaments.

Scalar complexity A way of measuring complexity with a geometric rational. (See *Primerr* for details.)

STD error The standard deviation of the weighted errors of a temperament.

Temperament A set of pitches used to approximate just intonation.

Temperament class Different tuned temperaments that share the same mapping from just intonation.

TOP-RMS error A Tenney-weighted, optimal, prime, root mean squared error of a temperament class. (See Section 2 of *Primerr* for details.)

10 Change Log

Considered finished. (2009-01-03)

Corrected an error copied across different equations, and some minor corrections. (2010-07-05)

Corrected spelling mistakes. (2010-09-25)

Fixed an obviously wrong date. (2016-09-14)